A more general categorical framework for congruence of applicative bisimilarity

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Abstract

We prove a general congruence result for bisimilarity in higher-order languages, which generalises previous work [7, 18] to languages specified by a labelled transition system in which programs may occur as labels, and which may rely on operations on terms other than capture-avoiding substitution. This is typically the case for PCF, λ-calculus with delimited continuations, and early-style bisimilarity in higher-order process calculi.

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1 Introduction

General congruence results for bisimilarity based on category theory date back at least to Turi and Plotkin's seminal paper [32], which covers labelled transition systems in a categorical version of the Positive GSOS format [6]. The result was then extended to languages with variable binding and renaming like the π-calculus [12, 31]. More recently, Borthelle et al. [7, 18] managed to deal with a wider class of languages, whose operational semantics may rely not only on renaming, but also on capture-avoiding substitution.

However, their result fails to cover significant languages to which Howe's method has been adapted, such as (variants of) PCF [17], λ-calculus with delimited continuations [9, 5], or (early-style) higher-order process calculi [30, 23]. The reason Borthelle et al.'s framework does not cover such applications is that they are specified by labelled transition systems

- in which programs may occur as labels, or
- which rely on operations on terms other than capture-avoiding substitution.

In this paper, we extend Borthelle et al.'s result to such languages, which requires a non-trivial extension of the proof method, essentially abstracting over ideas from Bernstein [4].

We introduce **algebraic transition systems**, which model transition systems whose vertices (=states) bear some algebraic structure, and which may have arbitrary vertices as labels. For such transition systems, we define **enhanced bisimilarity** as an abstract counterpart to applicative bisimilarity

We introduce **operational signatures**, which allow us to generate algebraic transition systems of interest, including all above-mentioned languages. Following initial-algebra semantics [15], an operational signature specifies algebraic structure and transition rules, and, in applications, the initial object in the category of models of an operational signature is the desired syntactic transition system.

Finally, we prove (Theorem 52) that, under suitable conditions, enhanced bisimilarity in the algebraic transition system generated by an operational signature is a congruence for the considered algebraic structure. We also exhibit (Theorem 61) sufficient conditions that are easier to check in practice. This covers all above-mentioned applications, except higher-order process calculi, whose operational signatures fail to satisfy the required conditions.

Related work

Beyond Borthelle et al. [18], which was discussed above, the most closely related work is Goncharov et al.'s [16] bialgebraic framework for higher-order operational semantics. They upgrade Turi and Plotkin's [32] original presentation of operational semantics as a natural transformation into a dinatural transformation, which allows them to cover transitions with programs as labels. Their main applications are strong variants of applicative bisimilarity for pure λ-calculus (call-by-name and call-by-value). In its current state, their framework cannot handle non-deterministic computation, hence in particular weak variants of bisimilarity.

Plan

We start in §2 with an overview of the development. In §3, we then present our running example, which will be used as the basis of our abstraction process. We then introduce our abstract notions of transition systems (§4), and algebraic transition systems (§5), together with bisimilarity and its enhanced variant. Finally, we introduce operational signatures and state our main results in §6, and conclude in §7.

Prerequisites and notations

We assume some basic knowledge of category theory [24], notably including factorisation systems and monad distributive laws [3]. Additionally, we rely in places on locally presentable categories [1], but this may be taken as technical, and ignored on a first reading. We often conflate natural numbers n with sets $\{1, \ldots, n\}$. We denote by $\mathbb{n}$ the corresponding ordinal viewed as a category, so that, e.g., $\mathscr{C}^2$ is the usual category of morphisms in $\mathscr{C}$. We let **CAT** denote the category of locally small categories. Moreover, we denote by $\widehat{\mathbb{C}}$ the category of (contravariant) presheaves over a given category $\mathbb{C}$, and by $\mathbf{y}\colon \mathbb{C} \to \widehat{\mathbb{C}}$ the Yoneda embedding. Furthermore, we recall that endofunctor algebras differ from monad algebras. (A monad algebra structure must be suitably compatible with unit and multiplication.) We write $F\text{-}\mathbf{alg}$ for endofunctor algebras, and $T\text{-}\mathbf{Alg}$ for monad algebras (capital 'A'!). Finally, for any endofunctor F on a sufficiently nice category, e.g., a presheaf category, we write F^* for the free monad on F, which is furthermore **algebraically free** in the sense that $F\text{-}\mathbf{alg} \cong F^*\text{-}\mathbf{Alg}$.

2 Overview

The development roughly follows [18]. We summarise it here, emphasising the differences. Our running example throughout is a pure λ-calculus with delimited continuations [5].

2.1 Transition systems

Let us first sketch our notion of transition system, starting from graphs. Consider the diagonal functor $\Delta\colon \mathbf{Set} \to \mathbf{Set}$, defined by $\Delta(X) = X^2$. A graph consists of two sets E and V, equipped with two maps $E \to V$, or equivalently a map $E \to \Delta(V)$.

Hirschowitz and Lafont [18] propose a "typed" generalisation: they postulate a category $\mathbb{VT}$ of **vertex types**, a category $\mathbb{ET}$ of **edge types**, and two functors $\mathbf{s}, \mathbf{t}\colon \mathbb{ET} \to \mathbb{VT}$ associating to each edge type the types of its source and target. A transition system in their sense consists of a **vertex object** V in $\widehat{\mathbb{VT}}$, a **edge object** $E \in \widehat{\mathbb{ET}}$, and a morphism

$E \to \Delta(V)$, where $\Delta\colon \widehat{\mathbb{VT}} \to \widehat{\mathbb{ET}}$ maps any V to $\Delta(V)(\alpha) = V(\mathbf{s}(\alpha)) \times V(\mathbf{t}(\alpha))$, for all $\alpha \in \mathbb{ET}$. Taking $\mathbb{VT} = \mathbb{ET} = 1$, one recovers plain graphs.

In this paper, in order to account for labels, we generalise this by adding a functor $\mathbf{l}$ associating to each edge type $\alpha \in \mathbb{ET}$ a sequence $\mathbf{l}(\alpha) = (\mathbf{l}^\alpha_1, \ldots, \mathbf{l}^\alpha_{n_\alpha})$ of vertex types. A tuple $(\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ is called a **Howe context**. Let us fix one for the rest of this section.

We modify Δ accordingly, defining it by

$$\Delta(V)(\alpha) \quad = \quad V(\mathbf{s}(\alpha)) \quad \times \quad \left(\prod_{i=1}^{n_\alpha} V(\mathbf{l}^\alpha_i)\right) \quad \times \quad V(\mathbf{t}(\alpha)).$$

A transition system again consists of objects $V \in \widehat{\mathbb{VT}}$ and $E \in \widehat{\mathbb{ET}}$, together with a morphism $E \to \Delta(V)$, which means that, to each edge, we associate a source, a target, and a sequence of labels of suitable types. For such transition systems, we define a generalisation of bisimulation, straightforwardly.

2.2 Algebraic transition systems and enhanced bisimilarity

Let us now briefly explain the notion of algebraic structure that we adopt. Following Fiore et al. [11, 13], Borthelle et al. [7, 18] use Σ-monoids, which are designed to model syntax with substitution. In this paper, relying on Hirschowitz and Lafont [19], we adopt a different notion of algebraic structure designed to cover syntax with more general additional operations.

▶ **Definition 1.** *An* ***enhanced syntax*** *(on $\widehat{\mathbb{VT}}$) consists of*

- *finitary functors $\Sigma\colon \widehat{\mathbb{VT}} \to \widehat{\mathbb{VT}}$ and $\Gamma\colon \widehat{\mathbb{VT}}^2 \to \widehat{\mathbb{VT}}$ such that Γ is* ***left-cocontinuous****, i.e., cocontinuous in its first argument, equipped with*
- *a distributive law $\delta\colon T \circ S \to S \circ T$, where $S = \Sigma^*$ denotes the monad freely generated by Σ and $T = \Gamma^*_S$ the one generated by $X \mapsto \Gamma(X, S(X))$.*

Here, Σ models basic syntax, and Γ models additional operations like substitution. The fact that Γ is a bifunctor is for distinguishing a "main" occurrence in its arity, which is used below in the definition of enhanced bisimilarity. The distributive law models commutation of additional operations with basic ones, at the main occurrence (typically $(M\ N)[\sigma] = M[\sigma]\ N[\sigma]$).

Following initial-algebra semantics [15], the main object of interest here is the initial Σ-algebra $S(\emptyset)$, and the main point is that it automatically possesses T-algebra structure, given by the composite $T(S(\emptyset)) \xrightarrow{\delta_\emptyset} S(T(\emptyset)) \cong S(\emptyset)$ (the initial object is a T-algebra by cocontinuity, hence $T(\emptyset) \cong \emptyset$). This algebra structure in fact makes $S(\emptyset)$ into an initial algebra for the composite monad ST.

Fixing some enhanced syntax $\sigma = (\Sigma, \Gamma, \delta)$, for us, an algebraic transition system is thus a transition system $E \to \Delta(V)$, equipped with ST-algebra structure on V. We call such transition systems **σ-algebraic**.

Finally, for any σ-algebraic transition system $G = (E, V, \partial)$, we define **enhanced bisimilarity**, denoted by $\sim^\sigma_G$, as the greatest bisimulation R which is **enhanced**, in the sense that $\Gamma(R, V) \subseteq R$ – this is where we use the fact that Γ is a bifunctor. In concrete instances, as noticed by Borthelle et al. [7, 18], enhanced bisimilarity agrees with applicative bisimilarity.

The goal is then to prove that, in algebraic transition systems G of interest, enhanced bisimilarity $\sim^\sigma_G$ is a **congruence**, i.e., $\Sigma(\sim^\sigma_G) \subseteq \sim^\sigma_G$.

2.3 Operational signatures

For this, we restrict attention to algebraic transition systems generated by a suitable notion of **operational signature**, which we now describe. Operational signatures comprise two components, one for generating an enhanced syntax, the other for specifying transition rules.

▶ **Definition 2.** *A **syntactic signature** is an endofunctor* Σ *equipped with a sequence*

$$T_0 = \mathrm{id} \xrightarrow{(\Gamma_1, d_1)} T_1 \quad \dots \quad T_{n-1} \xrightarrow{(\Gamma_n, d_n)} T_n \tag{1}$$

*of **incremental structural laws** [19]. An incremental structural law* $T \to T'$ *consists of a finitary, left-cocontinuous bifunctor* $\Gamma \colon \widehat{\mathbb{V}\mathbb{T}}^2 \to \widehat{\mathbb{V}\mathbb{T}}$*, together with a natural transformation* $d_{X,Y} : \Gamma(\Sigma(X), Y) \to S(T(\Gamma(X, S(T(Y))) + X + Y))$*, such that* $T' = T \oplus \Gamma_S^*$*, where* $\oplus$ *denotes monad coproduct.*

In examples, a natural transformation $d_{X,Y}$ amounts to a definition by structural recursion, where the first argument of Γ models the decreasing occurrence of the argument, and the second argument models other occurrences. Given any syntactic signature (1), the given incremental structural laws induce distributive laws $\delta_i \colon T_i \circ S \to S \circ T_i$, hence in particular $\delta_n \colon T_n \circ S \to S \circ T_n$, and furthermore we have $T_n = (\sum_i \Gamma_i)_S^*$. Thus, letting $\mathbf{d}$ denote the given syntactic signature, the triple $\sigma(\mathbf{d}) = (\Sigma, \sum_i \Gamma_i, \delta_n)$ forms an enhanced syntax. As a bonus, one can show that algebras for the composite monad ST_n are equivalently objects equipped with suitably coherent algebra structure for Σ and each functor $X \mapsto \Gamma_i(X, X)$.

The next step is to specify the dynamics of algebraic transition systems of interest. This is done by introducing **dynamic signatures**. Roughly, a dynamic signature over an enhanced syntax σ is an endofunctor on σ-algebraic transition systems, which is required to preserve the vertex object and satisfy a suitable "structuralness" condition inspired by structural operational semantics [26]. Intuitively, a dynamic signature Σ_1 is a family of transition rules, and structuralness demands that, in each transition rule, the source of the conclusion has depth at most one.

Pursuing the analogy, σ-algebraic transition systems satisfying the rules are a special kind of Σ_1-algebras which we call **vertical**. Verticality means that the algebra structure is trivial on vertices: this enforces that satisfying the rules is only about edges, not vertices.

Finally, an **operational signature** consists of a syntactic signature $\mathbf{d}$, and a dynamic signature Σ_1 on $\sigma(\mathbf{d})$. The real object of interest is here the initial vertical Σ_1-algebra, say $\mathbf{Z} = (E_{\mathbf{Z}}, V_{\mathbf{Z}}, \partial_{\mathbf{Z}})$, which in applications is the desired syntactic transition system.

2.4 Congruence of enhanced bisimilarity

Our goal is then to prove that, under suitable hypotheses, enhanced bisimilarity $\sim_{\mathbf{Z}}^{\sigma(\mathbf{d})}$ in the initial vertical Σ_1-algebra is a congruence. For this, abstracting over Bernstein's [4] proof, we start by defining **flexible bisimulation**, a variant of Sangiorgi's BA-bisimulation [29]. Flexible bisimulation is like plain bisimulation: given related elements e and e', any transition from e should be matched by some transition from e'. The difference is that, instead of having the same label, the matching transition should exist for any related label. Defining **functional flexible bisimulations** to be morphisms of algebraic transition systems whose graph is a flexible bisimulation, our main result (Theorem 52) states that if the dynamic signature Σ_1 preserves functional flexible bisimulations, then $\sim_{\mathbf{Z}}^{\sigma(\mathbf{d})}$ is a congruence.

Finally, preservation of functional flexible bisimulations is quite an abstract condition, so we set out to design a more concrete criterion for making the result easier to apply. In fact, if the considered dynamic signature Σ_1 is **familial** [10, 8, 33, 14], then preservation of functional flexible bisimulations becomes quite tractable, as we now explain. Following Joyal et al. [22], we first characterise functional flexible bisimulations as the right class of a weak factorisation system [20, 28] – we call the left class **cofibrations**. Furthermore, when the dynamic signature is familial, a transition rule with conclusion of type any α, is intuitively an element of $\Sigma_1(1)(\alpha)$, and we extract for each rule two algebraic transition systems A and B, and a morphism $\varphi \colon A \to B$, such that, intuitively, A describes the metavariables occurring in the source and label of the conclusion, B describes all metavariables in the rule, including transition premises, and φ embeds the former into the latter. We call φ the **border arity** of the rule. The main point is then that a familial Σ_1 preserves functional flexible bisimulations iff all border arities are cofibrations (Theorem 61). How is this any more concrete? Well, cofibrations are well-known to be closed under composition and cobase change, so in order to check preservation of functional flexible bisimulations, it suffices to reconstruct the border arity of each rule from generating cofibrations, by composition and cobase change. This reconstruction process is close in spirit to usual acyclicity criteria [21, 4].

▶ **Example 3.** Taking algebraic transition systems to be just plain graphs, for a rule like $\dfrac{a \to b \qquad b \to c}{a \to c}$, A would be the one-vertex graph, B would consist of two composable edges $x \to y \to z$, and φ would pick x. To check that it is a cofibration, we reconstruct it as the bottom composite in

$$\begin{array}{ccccc} & & [0] & \xrightarrow{s} & [1] \\ & & \downarrow & & \downarrow \\ A = [0] & \xrightarrow{s} & [1] & \longrightarrow & B. \end{array}$$

As an application, we recover congruence of applicative bisimilarity in the considered λ-calculus with delimited continuations [5].

3 A concrete example

As a concrete example result that we want to abstract over, let us recall the case of λ-calculus with delimited continuations. We present it in a non-standard way in order for it to fit the abstract framework. Indeed, the framework is based on **structural** operational semantics [26], in the sense that, in each transition rule, the source of the conclusion has depth at most one. Following [7, 18], we also present the definition of the open extension of applicative bisimilarity to make it compatible with the abstract developments to come.

The syntax, presented in the usual, informal way, is as below left,

$$\text{Values} \ni v ::= x \mid \lambda x.e \qquad\qquad \square[e] = e \qquad (2)$$

$$\text{Programs} \ni e ::= v \mid e_1\ e_2 \mid \mathcal{S}x.e \mid \langle e \rangle \qquad\qquad (v\ E)[e] = v\ E[e] \qquad (3)$$

$$\text{Evaluation contexts} \ni E ::= \square \mid E\ e \mid v\ E \qquad\qquad (E\ e')[e] = E[e]\ e'. \qquad (4)$$

where x binds in e, in both $\lambda x.e$ and $\mathcal{S}x.e$. Capture-avoiding substitution and context application are defined as usual. E.g., context application is defined as above right. The dynamics are governed by the rules in Figure 1. There are three kinds of transitions, of types $e \xrightarrow{\tau} e'$, $e \xrightarrow{v} e'$, $e \xrightarrow{E} e'$, where all expressions are closed. The first four rules deal with functions. The first two rules suffice to make (β) derivable, as shown in Figure 1. The next two rules are the usual context rules. The last three rules, where α ranges over all labels, enforce that we work with weak bisimulation: we close transitions under composition with silent transitions. The remaining rules describe the dynamics of $\mathcal{S}x.e$ and $\langle e \rangle$, which are

$$\frac{e_1 \xrightarrow{v} e_2}{e_1\ v \xrightarrow{\tau} e_2}\ (\beta') \qquad \frac{}{\lambda x.e \xrightarrow{v} e[x \mapsto v]} \qquad \frac{e_1 \xrightarrow{\tau} e_1'}{e_1\ e_2 \xrightarrow{\tau} e_1'\ e_2} \qquad \frac{e_2 \xrightarrow{\tau} e_2'}{v\ e_2 \xrightarrow{\tau} v\ e_2'} \qquad \frac{}{\langle v \rangle \xrightarrow{\tau} v}$$

$$\frac{e \xrightarrow{\tau} e'}{\langle e \rangle \xrightarrow{\tau} \langle e' \rangle} \qquad \frac{e \xrightarrow{\square} e'}{\langle e \rangle \xrightarrow{\tau} e'} \qquad \frac{e_1 \xrightarrow{E[\square\ e_2]} e_3}{e_1\ e_2 \xrightarrow{E} e_3} \qquad \overset{\text{(SA)}}{\frac{e_1 \xrightarrow{E[v\ \square]} e_2}{v\ e_1 \xrightarrow{E} e_2}} \qquad \frac{}{\mathcal{S}k.e \xrightarrow{E} \langle e[k \mapsto \lambda x.\langle E[x] \rangle] \rangle}$$

$$\frac{}{e \xrightarrow{\tau} e} \qquad \frac{e_1 \xrightarrow{\tau} e_2 \xrightarrow{\alpha} e_3}{e_1 \xrightarrow{\alpha} e_3} \qquad \frac{e_1 \xrightarrow{\alpha} e_2 \xrightarrow{\tau} e_3}{e_1 \xrightarrow{\alpha} e_3} \qquad \text{Deriving } \beta\text{:}\ \frac{\frac{}{\lambda x.e \xrightarrow{v} e[x \mapsto v]}}{(\lambda x.e)\ v \xrightarrow{\tau} e[x \mapsto v]}\ (\beta')$$

Figure 1 Transition rules

respectively called **shift** and **reset**. The first two of them enforce that silent computation occurs normally inside any reset, and if it succeeds, i.e., if it results in a value, then the reset disappears. The next rules describe how shift captures the ambient context up to the enclosing reset, say E, and substitutes its reification $\lambda k.\langle E[k] \rangle$ as a value for the bound variable, placing a new reset around the result.

Bisimulation is then as expected:

▶ **Definition 4.** *A binary relation R between closed programs is a **simulation** iff for all $e\ R\ e'$ and transitions $e \xrightarrow{\alpha} e_1$, there exists a transition $e' \xrightarrow{\alpha} e_1'$ such that $e_1\ R\ e_1'$. A **bisimulation** is a simulation whose converse relation also is a simulation.*

▶ **Definition 5** ([7, 18])**.** *A relation R on potentially open expressions is **enhanced** iff it is closed under substitution, context composition, and context application, i.e., $a\ R\ a'$ entails $a[\sigma]\ R\ a'[\sigma]$ for all substitutions σ, $E\ R\ E'$ entails $E[e]\ R\ E'[e]$ and $E[E'']\ R\ E'[E'']$, for all e and E''.*

*An **enhanced bisimulation** is an enhanced relation R whose restriction to closed programs is a bisimulation.*

▶ **Proposition 6.** *There is a largest enhanced bisimulation, called **applicative bisimilarity**.*

The result that we want to abstract over is:

▶ **Theorem 7** (generalised variant of [5, Theorem 1])**.** *Applicative bisimilarity is a **congruence**, in the sense that it is preserved by all constructions of the language.*

▶ Remark 8. It is not entirely trivial that this agrees with Biernacki and Lenglet's presentation. In fact, their transition system only differs in that they replace rule (β') with the standard rule (β). We have already seen that (β) is derivable from (β'), and conversely (β') is admissible in their transition system. Indeed, suppose given any transition $e_1 \xrightarrow{v} e_2$. By an easy induction, there exist transitions $e_1 \xrightarrow{\tau} \lambda x.e_3 \xrightarrow{v} e_3[x \mapsto v] \xrightarrow{\tau} e_2$. Hence, grouping saturation rules, we derive (β) as follows.

$$\frac{\dfrac{e_1 \xrightarrow{\tau} \lambda x.e_3}{e_1\ v \xrightarrow{\tau} (\lambda x.e_3)\ v} \qquad (\lambda x.e_3)\ v \xrightarrow{\tau} e_3[x \mapsto v] \qquad e_3[x \mapsto v] \xrightarrow{\tau} e_2}{e_1\ v \xrightarrow{\tau} e_2}$$

Our problem is that this result is not an instance of Borthelle et al.'s [18, Theorem 6.15], because the dynamics rely on two features that are not handled:

(a) operations on terms, context application and composition, which differ from substitution,

(b) and contexts and values occurring as labels.

For (a), context application and composition might be encodable in Borthelle et al.'s setting, perhaps by resorting to the skew monoidal variant [7]. But this is quite artificial, and requires extra work that should not be necessary. For (b), it appears to be a hard obstruction.

4 Transition systems in the abstract

In this section, we start to abstract over the development of §3, by introducing a notion of labelled transition system, together with its associated notion of bisimilarity.

4.1 Howe contexts

Let us start by formally introducing Howe contexts, as sketched in §2.

▶ **Definition 9.** *A **Howe context** consists of*

- *a small category* $\mathbb{VT}$ *of **state types**,*
- *a small category* $\mathbb{ET}$ *of **transition types**,*
- ***source** and **target** functors* $\mathbf{s}, \mathbf{t} \colon \mathbb{ET} \to \mathbb{VT}$*, and*
- *a **label** functor* $\mathbf{l} \colon \mathbb{ET} \to \widehat{\mathbb{VT}}$*, such that each* $\mathbf{l}(c)$ *is a finite coproduct of representables.*

▶ **Example 10.** For plain graphs, we would take:

- $\mathbb{VT}$ to be the terminal category, because there is just one kind of vertex,
- $\mathbb{ET}$ to also be the terminal category, because there is just one kind of edge,
- the source and target functors both are the unique functor $1 \to 1$, and
- the label functor to map the unique object to the empty coproduct, i.e., $\emptyset$.

▶ **Example 11.** For modelling the transition system of §3, we need a presheaf on $\mathbb{VT}$ to be equivalent to a triple of functors $V_\mathbf{p}, V_\mathbf{v}, V_\mathbf{c} \colon \mathbb{F} \to \mathbf{Set}$, where $\mathbb{F}$ denotes a skeleton of the category of finite sets, e.g., finite ordinals and all maps between them, equipped with a natural transformation $\iota \colon V_\mathbf{v} \to V_\mathbf{p}$, or otherwise said to a functor $\mathbb{F} \to \mathbf{Set}^{1+2}$. We think of $V_\mathbf{p}(n)$, $V_\mathbf{v}(n)$, and $V_\mathbf{c}(n)$ as sets of programs, values, and contexts with n free variables, respectively. For making this into a presheaf category, let us first observe that such tuples $(V_\mathbf{p}, V_\mathbf{v}, V_\mathbf{c}, \iota)$ are precisely the objects of the oplax limit of the functor $\Delta_{\mathbf{y}in_1} \colon \widehat{\mathbb{F}^{op} + \mathbb{F}^{op}} \to \widehat{\mathbb{F}^{op}}$ mapping any copairing $[V_\mathbf{p}, V_\mathbf{c}]$ to $V_\mathbf{p}$. But, as we now recall, oplax limits of this form are equivalent to presheaf categories.

▶ **Definition 12.** *For any small categories* $\mathbb{X}$ *and* $\mathbb{Y}$*, and functor* $F \colon \mathbb{X} \to \widehat{\mathbb{Y}}$*, the **collage** of* F*, denoted by* $\mathbb{Y}[\mathbb{X}]_F$*, or merely* $\mathbb{Y}[\mathbb{X}]$ *when* F *is clear from context, has as objects the disjoint union of those of* $\mathbb{X}$ *and* $\mathbb{Y}$*, and morphisms defined by cases as follows.*

$$\begin{array}{rclcrcl} \mathbb{X}[\mathbb{Y}](x,x') & = & \mathbb{X}(x,x') & \qquad & \mathbb{X}[\mathbb{Y}](y,x) & = & F(x)(y) \\ \mathbb{X}[\mathbb{Y}](y,y') & = & \mathbb{Y}(y,y') & & \mathbb{X}[\mathbb{Y}](x,y) & = & \emptyset \end{array}$$

Composition is defined as in $\mathbb{X}$ *and* $\mathbb{Y}$ *in both left-hand cases, and otherwise by action of* F*.*

▶ **Proposition 13** ([8, Lemma 4.9])**.** *For any small categories* $\mathbb{X}$ *and* $\mathbb{Y}$*, and functor* $F \colon \mathbb{X} \to \widehat{\mathbb{Y}}$*, letting* $\Delta_F(Y)(x) = \widehat{\mathbb{Y}}(F(x), Y)$ *denote the induced **nerve** functor* $\widehat{\mathbb{Y}} \to \widehat{\mathbb{X}}$*, the oplax limit* $\widehat{\mathbb{X}}/\Delta_F$ *is equivalent to the category* $\widehat{\mathbb{Y}[\mathbb{X}]}$ *of presheaves on the collage of* F*.*

Now, the above functor $\Delta_{\mathbf{y}in_1}$ is indeed the nerve of $\mathbb{F}^{op} \xrightarrow{in_1} \mathbb{F}^{op} + \mathbb{F}^{op} \xrightarrow{\mathbf{y}} \widehat{\mathbb{F}^{op} + \mathbb{F}^{op}}$ since we have $\Delta_{\mathbf{y}in_1}[V_\mathbf{p}, V_\mathbf{c}](n) = V_\mathbf{p}(n) = [V_\mathbf{p}, V_\mathbf{c}](in_1(n)) = \widehat{\mathbb{F}^{op} + \mathbb{F}^{op}}(\mathbf{y}(in_1(n)), [V_\mathbf{p}, V_\mathbf{c}])$. We obtain:

▶ **Corollary 14.** *Letting* $\mathbb{VT} = (\mathbb{F}^{op} + \mathbb{F}^{op})[\mathbb{F}^{op}]_{\mathbf{y}in_1}$*, we have* $[\mathbb{F}, \mathbf{Set}^{1+2}] \simeq \widehat{\mathbb{VT}}$*.*

▶ **Notation 1.** *We denote objects* $in_1 n$*,* $in_2 n$*, and* $in_3 n$ *of* $\mathbb{VT}$ *by* $n_\mathbf{v}$*,* $n_\mathbf{p}$*,* $n_\mathbf{c}$*, respectively, for values, programs, and contexts. For any* $V \in \widehat{\mathbb{VT}}$*, we denote the corresponding functors* $\mathbb{F} \to \mathbf{Set}$ *by* $V_\mathbf{v}$*,* $V_\mathbf{p}$*, and* $V_\mathbf{c}$*, so that, e.g.,* $V(n_\mathbf{v}) = V_\mathbf{v}(n)$*.*

Let us now define $\mathbb{ET} = 3 = \{[\tau], [\mathbf{v}], [\mathbf{c}]\}$, where $[\alpha]$ indicates a label of type α. Accordingly, writing $c : a \xrightarrow{L} b$ for $\mathbf{s}(c) = a$, $\mathbf{l}(c) = L$, and $\mathbf{t}(c) = b$, and respectively interpreting τ, $\mathbf{v}$, and $\mathbf{c}$ as $\emptyset$, $\mathbf{y}_\mathbf{v}$, and $\mathbf{y}_\mathbf{c}$, we put: $[\alpha] : 0_\mathbf{p} \xrightarrow{\alpha} 0_\mathbf{p}$, for all $\alpha \in \{\tau, \mathbf{v}, \mathbf{c}\}$.

4.2 Generalised transition systems

Let us now introduce transition systems. Let us fix a Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ for the whole subsection, and start by relating both categories $\widehat{\mathbb{VT}}$ and $\widehat{\mathbb{ET}}$.

▶ **Definition 15.** *We define four functors* $\widehat{\mathbb{VT}} \to \widehat{\mathbb{ET}}$ *as follows, for all* $V \in \widehat{\mathbb{VT}}$ *and* $\alpha \in \mathbb{ET}$*.*

$$\begin{array}{rclcrcl} \Delta_\mathbf{s}(V)(\alpha) & = & V(\mathbf{s}(\alpha)) & \qquad & \Delta_\mathbf{l}(V)(\alpha) & = & \widehat{\mathbb{VT}}(\mathbf{l}(\alpha), V) \\ \Delta_\mathbf{t}(V)(\alpha) & = & V(\mathbf{t}(\alpha)) & & \Delta_\mathbb{H}(V) & = & \Delta_\mathbf{s}(V) \times \Delta_\mathbf{l}(V) \times \Delta_\mathbf{t}(V), \end{array}$$

▶ **Notation 2.** *We often abbreviate* $\Delta_\mathbb{H}$ *to* Δ *when* $\mathbb{H}$ *is clear from context. We also use juxtaposition of indices to denote product of the corresponding functors, e.g.,* $\Delta_{\mathbf{s},\mathbf{l}} := \Delta_\mathbf{s} \times \Delta_\mathbf{l}$*.*

▶ **Definition 16.** *An* $\mathbb{H}$***-transition system*** G *consists of a* ***vertex*** *presheaf* $V_G \in \widehat{\mathbb{VT}}$*, an* ***edge*** *presheaf* $E_G \in \widehat{\mathbb{ET}}$*, and a* ***border*** *natural transformation* $\partial_G : E_G \to \Delta(V_G)$*.*

▶ Remark 17. Letting $\mathbf{l}(\alpha) = \sum_{i \in n_\alpha} \mathbf{y}_{\mathbf{l}_i^\alpha}$, we have $\Delta_\mathbf{l}(V)(\alpha) = [\sum_{i \in n_\alpha} \mathbf{y}_{\mathbf{l}_i^\alpha}, V] \cong \prod_{i \in n_\alpha} V(\mathbf{l}_i^\alpha)$ for any $\alpha \in \mathbb{ET}$ and $V \in \widehat{\mathbb{VT}}$. The border natural transformation thus has type

$$E(\alpha) \to V(\mathbf{s}(\alpha)) \times (\textstyle\prod_{i \in n_\alpha} V(\mathbf{l}_i^\alpha)) \times V(\mathbf{t}(\alpha)).$$

▶ **Example 18.** Let us unfold the definition for the Howe context of Example 11: a transition system consists of presheaves $V \in \widehat{\mathbb{VT}}$ and $E \in \widehat{\mathbb{ET}}$, equipped with maps

$$E[\tau] \to V_\mathbf{p}(0)^2 \qquad E[\mathbf{v}] \to V_\mathbf{p}(0) \times V_\mathbf{v}(0) \times V_\mathbf{p}(0) \qquad E[\mathbf{c}] \to V_\mathbf{p}(0) \times V_\mathbf{c}(0) \times V_\mathbf{p}(0).$$

We now equip $\mathbb{H}$-transition systems with morphisms:

▶ **Proposition 19.** $\mathbb{H}$*-transition systems are precisely the objects of the oplax limit category* $\widehat{\mathbb{ET}}/\Delta$ *of the functor* $\widehat{\mathbb{VT}} \xrightarrow{\Delta} \widehat{\mathbb{ET}}$ *in* $\mathbf{CAT}$*, or equivalently the comma category* $\mathrm{id}_{\widehat{\mathbb{ET}}} \downarrow \Delta$*.*

Proof. An object of the oplax limit is by definition a triple $(E, V, \partial : E \to \Delta(V))$. ◀

▶ **Definition 20.** *Let* $\mathbb{H}$*-*$\mathbf{Trans} = \widehat{\mathbb{ET}}/\Delta_\mathbb{H}$*.*

4.3 Bisimulation and bisimilarity

We now want to define bisimulation and bisimilarity, for any fixed Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$. Let us start with the notion of simulation.

▶ **Notation 3.** *A* ***span*** *is a pair of morphisms with the same source. In a category with binary products, we often write spans* $X \leftarrow R \rightarrow Y$ *as their pairings* $R \rightarrow X \times Y$*. The* ***converse*** *of a span* $\langle f, g\rangle : R \rightarrow X \times Y$ *is the composite* $\langle g, f\rangle : R \rightarrow X \times Y$*.*

In a presheaf category $\widehat{\mathbb{C}}$*, for any span* $j : R \rightarrow X \times Y$*, object* $c \in \mathbb{C}$*, and element* $r \in R(c)$*, we write* $r : x \; R \; y$ *when* $j_c(r) = (x, y)$*. We call* r *a* ***witness*** *that* x *and* y *are related by* R*.*

Finally, in any $\mathbb{H}$*-transition system* G*, for any transition type* α *with* $\mathbf{l}(\alpha) \cong \sum_{i \in n_\alpha} \mathbf{y}_{\mathbf{l}_i^\alpha}$*, we write* $e : x \xrightarrow{\alpha(l_1,\ldots,l_{n_\alpha})} y$ *to mean that* $e \in E_G(\alpha)$ *and* $\partial_G(e) = (x, (l_1, \ldots, l_{n_\alpha}), y)$*.*

▶ **Definition 21.** *For any* $\mathbb{H}$*-transition system* $G = (V, E, \partial : E \rightarrow \Delta V)$*, a given span* $j : R \rightarrow V^2$ *is a* ***simulation*** *when, for any transition* $e : x \xrightarrow{\alpha(l_1,\ldots,l_{n_\alpha})} x'$ *and witness* $r : x \; R \; y$*, there exists a transition* $f : y \xrightarrow{\alpha(l_1,\ldots,l_{n_\alpha})} y'$ *and a witness* $r' : x' \; R \; y'$*, as in*

$$\begin{array}{ccccc} x & & R(\mathbf{s}(\alpha)) & & y \\ {\scriptstyle e\,:\,\alpha(l_1,\ldots,l_{n_\alpha})}\downarrow & & & & \downarrow{\scriptstyle f\,:\,\alpha(l_1,\ldots,l_{n_\alpha})} \\ x' & & R(\mathbf{t}(\alpha)) & & y'. \end{array} \tag{5}$$

A span is a ***bisimulation*** *when it is a simulation and so is its converse. A* ***bisimulation relation*** *is a bisimulation which is also a relation, i.e., a mono* $R \hookrightarrow V^2$*.*

▶ **Proposition 22.** *The full subcategory* $\mathbf{Bisim}(G)$ *of* $\widehat{\mathbb{VT}}/V^2$ *spanning bisimulations admits a terminal object, which we call* ***bisimilarity*** *and denote by* $\sim_G$*.*

Proof. Bisimulation relations are stable under unions, so that a terminal object is given by the union of them all. ◀

5 Algebraic transition systems

In this section, we explain enhanced syntax, algebraic transition systems, and enhanced bisimulation in a bit more detail than in §2.2. The notion of enhanced syntax has already been introduced (Definition 1), and we fix a Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax $\sigma = (\Sigma, \Gamma, \delta : TS \rightarrow ST)$, where, we recall, $S = \Sigma^*$ and $T = \Gamma_S^*$.

5.1 Enhanced syntax

▶ **Definition 23.** *We call* ST*-algebras* ***σ-algebras*** *for short, and let* $\sigma\text{-}\mathbf{Alg} = ST\text{-}\mathbf{Alg}$*.*

▶ **Proposition 24.** *The initial* Σ*-algebra* $S\emptyset$ *is automatically a* T*-algebra, with structure map* $TS\emptyset \xrightarrow{\delta_\emptyset} ST\emptyset \xrightarrow{\cong} S\emptyset$*.*

Proof. By cocontinuity, $\emptyset$ is a Γ_S-algebra: we have $\Gamma(\emptyset, S(\emptyset)) \cong \emptyset$. It is thus an initial Γ_S-algebra, hence an initial T-algebra since $\Gamma_S\text{-}\mathbf{alg} \cong T\text{-}\mathbf{Alg}$. ◀

▶ **Example 25.** Following up on Example 11, the syntax and additional operations of §3 may be presented by an incremental structural law on $\widehat{\mathbb{VT}}$, as follows. First, basic operations are specified by the endofunctor Σ_0 defined as follows (recalling original notation on the right).

$\Sigma_0(X)_\mathsf{v}(n) = n + X_\mathsf{p}(n+1)$	$v ::= x \mid \lambda x.e$
$\Sigma_0(X)_\mathsf{p}(n) = \Sigma_0(X)_\mathsf{v}(n) + X_\mathsf{v}(n) + X_\mathsf{p}(n)^2 + X_\mathsf{p}(n+1) + X_\mathsf{p}(n)$	$e ::= v \mid e_1 \; e_2 \mid \mathcal{S}x.e \mid \langle e\rangle$
$\Sigma_0(X)_\mathsf{c}(n) = 1 + X_\mathsf{v}(n) \times X_\mathsf{c}(n) + X_\mathsf{c}(n) \times X_\mathsf{p}(n)$	$E ::= \square \mid E \; e \mid v \; E$

We then want to define the arity of additional operations, namely substitution, context application, and context composition. Since these three additional operations are independent, we may specify them at once by the bifunctor $\Gamma : \widehat{\mathbb{VT}}^2 \rightarrow \widehat{\mathbb{VT}}$ defined as follows.

$$\begin{array}{rcl} \Gamma(X,Y)_{\mathbf{v}}(n) & = & \sum_{m\in\mathbb{N}} X_{\mathbf{v}}(m) \times Y_{\mathbf{v}}(n)^m \\ \Gamma(X,Y)_{\mathbf{p}}(n) & = & \sum_{m\in\mathbb{N}} X_{\mathbf{p}}(m) \times Y_{\mathbf{v}}(n)^m + X_{\mathbf{c}}(n) \times Y_{\mathbf{p}}(n) \\ \Gamma(X,Y)_{\mathbf{c}}(n) & = & X_{\mathbf{c}}(n) \times Y_{\mathbf{c}}(n) \end{array} \qquad \begin{array}{rcl} v & +::= & v[\sigma] \\ e & +::= & e[\sigma] \mid E[e] \\ E & +::= & E[E'] \end{array}$$

That the actual definition of additional operations induces a distributive law of Γ_S^* over Σ^* is harder to see, and will follow from the theory of syntactic signatures below (Example 33).

5.2 Algebraic transition systems

Let us now introduce algebraic transition systems.

▶ **Definition 26.** *A **σ-transition system** is an $\mathbb{H}$-transition systems equipped with σ-algebra structure on its vertex object. A σ-transition system morphism is a morphism of $\mathbb{H}$-transition systems whose vertex component is a σ-algebra morphism. Let σ-**Trans** denote the category of σ-transition systems and morphisms between them.*

▶ **Proposition 27.** *The forgetful functor $\mathscr{U}$ has a left adjoint, say $\mathscr{L}\colon \mathbb{H}\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}$.*

Proof. The left adjoint maps any $\partial\colon E \to \Delta(V)$ to $E \xrightarrow{\partial} \Delta(V) \xrightarrow{\Delta(\eta_V^{ST})} \Delta(S(T(V)))$. ◀

We conclude this section by defining the notion of congruence.

▶ **Definition 28.** *For any σ-transition system $G = (V, E, \partial)$, a **congruence** is a span $R \to V^2$ for which there exists a morphism $\Sigma(R) \to R$ making the first diagram of Figure 2 commute.*

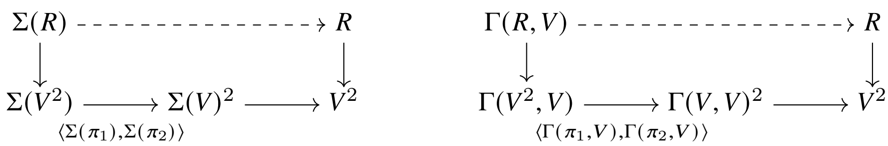

Figure 2 Congruence and enhancement

5.3 Enhanced bisimilarity

▶ **Definition 29.** *For any σ-algebra V, a span $p\colon R \to V^2$ is **enhanced** when there exists a morphism $\Gamma(R,V) \to R$ making the second diagram of Figure 2 commute.*

▶ **Definition 30.** *For any σ-transition system G, let $\mathbf{Bisim}^\sigma(G)$ denote the full subcategory of $\mathbf{Bisim}(G)$ on enhanced spans. We call such spans **enhanced bisimulations**.*

▶ **Proposition 31.** *For any σ-transition system G, $\mathbf{Bisim}^\sigma(G)$ admits a terminal object, which we call **enhanced** bisimilarity and denote by $\sim_G^\sigma$.*

Proof. Similar to Proposition 22, using left-cocontinuity of Γ. ◀

▶ **Example 32.** In the setting of Example 25, enhanced bisimilarity is applicative bisimilarity.

6 Signatures for operational semantics

6.1 Syntactic signatures for enhanced syntax

Syntactic signatures have already been introduced in Definition 2.

▶ **Example 33.** Following up on Example 25, the syntax and additional operations of §3 may be presented as an incremental structural law $d_{X,Y} : \Gamma_Y(\Sigma(X)) \to S(\Gamma_{S(Y)}(X) + X + Y)$ (taking $T_1 = \mathrm{id}$) on $\widehat{\mathbb{V}\mathbb{T}}$, as follows. For context application, Equations (2)–(4) may be interpreted as the component $\Sigma(X)_{\mathbf{c}}(n) \times Y_{\mathbf{p}}(n) \to S(\Gamma_{S(Y)}(X) + X + Y)_{\mathbf{p}}(n)$, namely we take them to mean

$$\begin{array}{rcl} (in_1(\star), y) & \mapsto & in'_3(y) \\ (in_2(v, E), y) & \mapsto & \iota(in'_2(v))\ in'_1(E, y) \\ (in_3(E, e), y) & \mapsto & in'_1(E, y)\ in'_2(e), \end{array}$$

where $in'_i = \eta^S \circ in_i$. For context composition, we define the component at $\mathbf{c}$ (for any n), by the exact same formulas, only with $y \in Y_{\mathbf{c}}(n)$. Substitution is defined similarly [11, 7, 18].

▶ **Proposition 34.** *For any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$ as in (1), the given incremental structural laws induce distributive laws $\delta_i : T_i \circ S \to S \circ T_i$, hence in particular $\delta_n : T_n \circ S \to S \circ T_n$, and furthermore we have $T_n = (\sum_i \Gamma_i)^*_S$. Thus, the triple $\sigma(\mathbf{d}) = (\Sigma, \sum_i \Gamma_i, \delta_n)$ forms an enhanced syntax.*

Proof. By [19, Theorem 4.2], each incremental structural law d_i induces a distributive law of $(T_{i-1} \oplus \Gamma^*_S)$ over S, i.e., of T_i over S by definition, using $(F + G)^* \cong F^* \oplus G^*$. ◀

Let us conclude this subsection by giving an explicit description of the algebras of the composite monad ST_n generated by a syntactic signature.

▶ **Definition 35.** *Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$. For $i \in n$, an* ***enhanced algebra*** *is an object equipped with algebra structures $a : \Sigma X \to X$, $b_1 : \Gamma_1(X, X) \to X$, ..., $b_n : \Gamma_n(X, X) \to X$ such that for all $i \in n$ the following diagram commutes,*

$$\begin{array}{ccccccc} \Gamma_i(\Sigma X, X) & \xrightarrow{(d_i)_{X,X}} & ST_i(\Gamma_i(X, ST_i X) + X + X) & \xrightarrow{ST_i(\Gamma_i(X, \bar{a} \circ S\bar{a}_i) + [X,X])} & ST_i(\Gamma_i(X, X) + X) & \xrightarrow{ST_i[b_i, X]} & ST_i X \\ \downarrow{\scriptstyle \Gamma_i(a,X)} & & & & & & \downarrow{\scriptstyle \bar{a} \circ S\bar{a}_i} \\ \Gamma_i(X, X) & & & \xrightarrow[b_i]{\qquad\qquad\qquad} & & & X \end{array}$$

where $\bar{a}_i : T_i X \to X$ and $\bar{a} : SX \to X$ denote the algebra structures induced by $(b_j)_{j<i}$, and a. Let $\mathbf{d}\text{-}\mathbf{Alg}$ denote the full subcategory of $(\Sigma + \sum_{i \in n} \Gamma_i)\text{-}\mathbf{alg}$ spanned by enhanced algebras.

▶ **Proposition 36.** *Let $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$ denote any syntactic signature. The forgetful functor $\sigma(\mathbf{d})\text{-}\mathbf{Alg} \to (\Sigma + \sum_{i \in n} \Gamma_i)\text{-}\mathbf{alg}$ lifts to $\mathbf{d}\text{-}\mathbf{Alg}$, and the lifting is an isomorphism. In short, we have $\sigma(\mathbf{d})\text{-}\mathbf{Alg} \cong \mathbf{d}\text{-}\mathbf{Alg}$ over $\widehat{\mathbb{C}}$.*

Proof. By induction on n and [19, Theorem 4.13]. ◀

6.2 Dynamic signatures

Let us now introduce signatures for the dynamical part of an operational semantics. We want a dynamic signature to be something like an endofunctor on σ-**Trans**, with built-in structuralness. For this, we introduce a variant of $\mathbb{H}$-transition systems called diplopic $\mathbb{H}$-transition systems, which feature an object of distinguished vertices, among which all sources of transitions must lie. This will enable structuralness, by allowing sources of conclusions of transition rules to have a distinguished head constructor. We fix an enhanced syntax $\sigma = (\Sigma, \Gamma, \delta : TS \to ST)$ for this subsection.

▶ **Definition 37.** *A **diplopic** $\mathbb{H}$**-transition system** G consists of a **vertex** object $V_G \in \widehat{\mathbb{VT}}$, a **distinguished vertex** object $D_G \in \widehat{\mathbb{VT}}$, an **edge** object $E_G \in \widehat{\mathbb{ET}}$, together with morphisms $\gamma_G \colon D_G \to V_G$ and $\partial_G \colon E_G \to \Delta_{\mathbf{s}}(D_G) \times \Delta_{\mathbf{l},\mathbf{t}}(V_G)$.*

*A **diplopic** σ**-transition system** is a diplopic $\mathbb{H}$-transition system G equipped with σ-algebra structure on V_G.*

As before, we organise both notions into categories $\mathbb{H}\text{-}\mathbf{Trans}_2 = \widehat{\mathbb{ET}}/\Delta_2$ and $\sigma\text{-}\mathbf{Trans}_2 = \mathbb{H}\text{-}\mathbf{Trans}_2 \times_{\widehat{\mathbb{V}}} \sigma\text{-}\mathbf{Alg}$, where Δ_2 denotes the composite $\widehat{\mathbb{VT}}^2 \xrightarrow{\langle \pi_1, \pi_2, \pi_2 \rangle} \widehat{\mathbb{VT}}^3 \xrightarrow{\Delta_{\mathbf{s}} \times \Delta_{\mathbf{l}} \times \Delta_{\mathbf{t}}} \widehat{\mathbb{ET}}$.

▶ **Definition 38.** *A **dynamic signature** over σ is a functor $\Sigma_1 \colon \sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}_2$ such that, for all $G \in \sigma\text{-}\mathbf{Trans}$, $V_{\Sigma_1(G)} = V_G$, $D_{\Sigma_1(G)} = V_G + \Sigma(V_G)$, and $\gamma_G \colon V_G + \Sigma(V_G) \to V_G$ is the canonical morphism (and similarly on morphisms).*

▶ **Example 39.** Letting $\mathbf{d}$ denote the syntactic signature of Example 33. The transition rules of §3 define a dynamic signature $\Sigma_1 \colon \sigma(\mathbf{d})\text{-}\mathbf{Trans} \to \sigma(\mathbf{d})\text{-}\mathbf{Trans}_2$. Its behaviour on the underlying $\sigma(\mathbf{d})$-algebra is fixed, so we merely need to define it on transitions. For any $G = (D, V, E, \partial) \in \sigma(\mathbf{d})\text{-}\mathbf{Trans}$ and $\alpha \in \mathbb{ET}$, we define $\Sigma_1(G)(\alpha)$ to be a coproduct over all rules ρ producing a transition of type α, of a set describing the premises of ρ. One non-trivial rule is (SA), whose set of premises is $E[\mathbf{c}] \times_{V(0_{\mathbf{c}})} (V(0_{\mathbf{c}}) \times V(0_{\mathbf{v}}))$. Concretely, it is the set of tuples $(r, (E, v))$, where r is a transition $e_1 \xrightarrow{E'} e_2$, and the pullback condition imposes $E' = E[v\ \square]$. (We take the pullback of $E[\mathbf{c}] \xrightarrow{\pi_2 \partial} V(0_{\mathbf{c}}) \xleftarrow{E[v\ \square] \hookleftarrow E, v} V(0_{\mathbf{c}}) \times V(0_{\mathbf{v}})$.) We define the source of $(r, (E, v))$ to be $in_2(in_2(\iota(v), e_1)) \in V_{\mathbf{p}}(0) + \Sigma_0(V)_{\mathbf{p}}(0)$, (i.e., recalling Σ_0 from Example 25, the formal application $\iota(v)\ e_1$,) its label to be E, and its target to be e_2.

Returning to the abstract setting, let us now define the category of models of a dynamic signature Σ_1. For this, we need to build an endofunctor out of Σ_1, hence a link between $\sigma\text{-}\mathbf{Trans}$ and $\sigma\text{-}\mathbf{Trans}_2$.

▶ **Definition 40.** *Let $\iota\text{-}\mathbf{Trans} \colon \sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}_2$ map any $E \to \Delta(V)$ to itself (with underlying arrow $V \to V$).*

▶ **Proposition 41.** *The functor $\iota\text{-}\mathbf{Trans} \colon \sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}_2$ is a (full) reflective embedding. The left adjoint, say $\rho\text{-}\mathbf{Trans} \colon \sigma\text{-}\mathbf{Trans}_2 \to \sigma\text{-}\mathbf{Trans}$ maps any $E \to \Delta_{\mathbf{s}}(D) \times \Delta_{\mathbf{l},\mathbf{t}}(V)$ to the composite $E \to \Delta_{\mathbf{s}}(D) \times \Delta_{\mathbf{l},\mathbf{t}}(V) \to \Delta(V)$.*

▶ **Definition 42.** *For any Σ_1, let $\check{\Sigma}_1$ be the composite $\sigma\text{-}\mathbf{Trans} \xrightarrow{\Sigma_1} \sigma\text{-}\mathbf{Trans}_2 \xrightarrow{\rho\text{-}\mathbf{Trans}} \sigma\text{-}\mathbf{Trans}$.*

Models of Σ_1 will almost be $\check{\Sigma}_1$-algebras. The problem is that a $\check{\Sigma}_1$-algebra structure includes in particular algebra structure for the action of $\check{\Sigma}_1$ on the underlying σ-algebra, i.e., algebra structure $V \to V$ for the identity endofunctor on σ-algebras. This structure is not relevant for our purposes, so we require it to be the canonical candidate, i.e., the identity on V.

▶ **Definition 43.** *A $\check{\Sigma}_1$-algebra structure $\check{\Sigma}_1(G) \to G$ is **vertical** when its image under the forgetful functor $\sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Alg}$ is the identity. A $\check{\Sigma}_1$-algebra is called **vertical** accordingly. Let $\check{\Sigma}_1\text{-}\mathbf{alg}_v$ denote the full subcategory of $\check{\Sigma}_1\text{-}\mathbf{alg}$ spanning all vertical algebras.*

▶ **Theorem 44.** *The forgetful functor $\check{\Sigma}_1\text{-}\mathbf{alg}_v \to \sigma\text{-}\mathbf{Trans}$ is monadic, and the initial $\check{\Sigma}_1$-algebra, say $\mathbf{Z}_{\Sigma_1}$, or $\mathbf{Z}$ for short when Σ_1 is clear from context, may be chosen to be vertical, hence in particular to also be initial in $\check{\Sigma}_1\text{-}\mathbf{alg}_v$. (In this case, $V_{\mathbf{Z}}$ is an initial σ-algebra.)*

Proof. Same as [18, Theorem 5.18 and Proposition 5.19]. ◀

▶ **Example 45.** For Σ_1 as in Example 39, $\mathbf{Z}$ is the syntactic transition system of §3.

Let us now collect the static and dynamic part of signatures and their models.

▶ **Definition 46.** *An **operational signature** consists of a syntactic signature* $\mathbf{d}$*, together with a dynamic signature* $\Sigma_1 \colon \sigma(\mathbf{d})\text{-}\mathbf{Trans} \to \sigma(\mathbf{d})\text{-}\mathbf{Trans}_2$ *over the generated enhanced syntax* $\sigma(\mathbf{d})$ *(Proposition 34). The category of* $(\mathbf{d}, \Sigma_1)$*-algebras is* $\check{\Sigma}_1\text{-}\mathbf{alg}_v$*.*

By definition, we have:

▶ **Proposition 47.** *The initial vertical* $\check{\Sigma}_1$*-algebra is an initial* $(\mathbf{d}, \Sigma_1)$*-algebra.*

6.3 Congruence of enhanced bisimilarity

In this subsection, we state our main congruence result. For this, we need to make an important hypothesis involving so-called functional flexible bisimulations. These are like a functional version of bisimulations, where labels are required to be related instead of identical, much as in Sangiorgi's BA-bisimulation [29], which we need to define both for algebraic transition systems and their diplopic variant. The hypothesis will then require that the considered dynamic signature Σ_1 preserve functional flexible bisimulations. We again fix a Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax σ over it.

▶ **Definition 48.** *A morphism* $f \colon R \to X$ *in* $\mathbb{H}\text{-}\mathbf{Trans}_2$ *is a **functional flexible bisimulation** iff for any* $\alpha \in \mathbb{ET}$*,* $r \in D_R(\mathbf{s}(\alpha))$*,* $(r_1, \ldots, r_{n_\alpha}) \in \Delta_{\mathbf{l}}(V_R)(\alpha)$*, and transition* $e' \colon f_D(r) \xrightarrow{\alpha(f_V(r_1), \ldots, f_V(r_{n_\alpha}))} x'$ *there exists* $e \colon r \xrightarrow{\alpha(r_1, \ldots, r_{n_\alpha})} r'$ *such that* $f_E(e) = e'$*.*

A morphism in $\sigma\text{-}\mathbf{Trans}_2$ *is a functional flexible bisimulation iff the underlying morphism in* $\mathbb{H}\text{-}\mathbf{Trans}_2$ *is. A morphism in* $\sigma\text{-}\mathbf{Trans}$ *is a functional flexible bisimulation iff its embedding into* $\sigma\text{-}\mathbf{Trans}_2$ *(by* $\iota\text{-}\mathbf{Trans}$*) is. In any of these categories* $\mathscr{C}$*, let* $\mathbf{FFBisim}(\mathscr{C})$ *denote the class of all functional flexible bisimulations.*

▶ **Definition 49.** *A dynamic signature* $\Sigma_1 \colon \sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}_2$ ***preserves functional flexible bisimulations** iff for all morphisms* f *in* $\sigma\text{-}\mathbf{Trans}$*, if* f *is a functional flexible bisimulations, then so is* $\Sigma_1(f)$*.*

Let us introduce a last hypothesis before stating the main result:

▶ **Definition 50.** *A functor is **algebraic** iff it is finitary and preserves wide pullbacks and reflexive coequalisers. A syntactic signature* (Σ, Γ, δ) *is algebraic if the endofunctor* Σ *is.*

▶ Remark 51. Algebraicity is straightforward to verify in all our applications.

▶ **Theorem 52.** *For any operational signature* $(\mathbf{d}, \Sigma_1)$*, if* $\mathbf{d}$ *is algebraic and* Σ_1 *preserves functional flexible bisimulations, then enhanced bisimilarity on the initial vertical* $\check{\Sigma}_1$*-algebra is a congruence.*

Proof. See Appendix A. ◀

6.4 Preservation of functional flexible bisimulations

In this section, we exhibit a sufficient condition for a dynamic signature to preserve functional flexible bisimulations, slightly generalising [18, §7]. Fixing a Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax $\sigma = (\Sigma_0, \Gamma, \delta \colon TS \to ST)$ on $\widehat{\mathbb{VT}}$, we first characterise $\mathbb{H}\text{-}\mathbf{Trans}$ and $\mathbb{H}\text{-}\mathbf{Trans}_2$ as presheaf categories, which allows us to characterise functional flexible bisimulations as the right class of a weak factorisation system [20, 28] – we call the left class **cofibrations**. We then recall familial functors, and define the notion of rule of a dynamic signature Σ_1, and the **border arity** of any rule. We finally show that a familial Σ_1 preserves functional flexible bisimulations iff the border arities of all rules are cofibrations.

Let us characterise transitions systems as presheaves, recalling Definition 12:

▶ **Proposition 53.** *We have* $\mathbb{H}$-$\mathbf{Trans} \simeq \widehat{\mathbb{VT}[\mathbb{ET}]}_{\mathbf{y_s+l+y_t}}$, *where* $\mathbf{y_s} + \mathbf{l} + \mathbf{y_t} \colon \mathbb{ET} \to \widehat{\mathbb{VT}}$.

Proof. The functor $\Delta_{\mathbb{H}}$ is the nerve of $\mathbf{y_s} + \mathbf{l} + \mathbf{y_t}$, so we conclude by [8, Lemma 4.9]. ◀

Doing the same for $\mathbb{H}$-$\mathbf{Trans}_2$ leads to considering the functor $\mathbb{ET} \to \widehat{\mathbb{VT}}^2$ mapping any α to the arrow $\mathbf{y}_{\mathbf{s}(\alpha)} \to \mathbf{y}_{\mathbf{s}(\alpha)} + \mathbf{l}(\alpha) + \mathbf{y}_{\mathbf{t}(\alpha)}$. But for [8, Lemma 4.9] to apply, we need the codomain of this functor to be a presheaf category. This is in fact the case up to equivalence:

▶ **Lemma 54.** *We have* $\widehat{\mathbb{VT}}^2 \simeq \widehat{\mathbb{VT}[\mathbb{VT}]}_{\mathbf{y}}$, *where* $\mathbf{y} \colon \mathbb{VT} \to \widehat{\mathbb{VT}}$.

▶ **Notation 4.** *For each state type $b \in \mathbb{VT}$, the category $\mathbb{VT}[\mathbb{VT}]$ has an object b_V corresponding to the vertex object, an object b_D for the distinguished vertex object, and a morphism $b_V \to b_D$.*

Gluing along the obtained functor $\alpha \mapsto \mathbf{y}_{\mathbf{s}(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(\mathbf{l}_i^\alpha)_V} + \mathbf{y}_{\mathbf{t}(\alpha)_V}$, we obtain:

▶ **Proposition 55.** *We have* $\mathbb{H}$-$\mathbf{Trans}_2 \simeq \widehat{\mathbb{VT}[\mathbb{VT}][\mathbb{ET}]}$.

Let us now characterise functional flexible bisimulations by a lifting property.

▶ **Definition 56.** *In a category $\mathscr{C}$, given a class $\mathbb{J}$ of morphisms, let $\mathbb{J}^{\pitchfork}$ consist of morphisms $f \colon X \to Y$ such that for any $j \colon A \to B$ in $\mathbb{J}$, any $(u, v) \colon j \to f$ in $\mathscr{C}^2$ admits a **lifting**, i.e., a morphism $k \colon B \to X$ such that $k \circ j = u$ and $f \circ k = v$. Let ${}^{\pitchfork}\mathbb{J}$ consist of all f such that any $(u, v) \colon f \to j$ in $\mathscr{C}^2$ admits a lifting. A $\mathbb{J}$-**cofibration** is an element of ${}^{\pitchfork}(\mathbb{J}^{\pitchfork})$.*

▶ **Proposition 57.** *For any $\mathbb{J}$,$\mathbb{J}$-cofibrations are closed under cobase change and composition.*

For any $\alpha \in \mathbb{ET}$, the element $(in_1(\mathrm{id}_{\mathbf{s}(\alpha)})) \in (\mathbf{y_s} + \mathbf{l} + \mathbf{y_t})(\alpha)(\mathbf{s}(\alpha))$, corresponds to a morphism $s_\alpha \colon \mathbf{s}(\alpha) \to \alpha$ in $\mathbb{VT}[\mathbb{ET}]$, and similarly we get morphisms $l_i^\alpha \colon \mathbf{l}_i^\alpha \to \alpha$ for all $i \in n_\alpha$.

▶ **Definition 58.** *Let $\mathbb{J}_\sigma$ denote the set of all maps $\mathscr{L}'(j_\alpha)$ in σ-$\mathbf{Trans}$, where $\mathscr{L}' \colon \widehat{\mathbb{VT}[\mathbb{ET}]} \to \sigma$-$\mathbf{Trans}$ is left adjoint to the forgetful functor, and $j_\alpha \colon \mathbf{y}_{\mathbf{s}(\alpha)} + \sum_{i \in n_\alpha} \mathbf{y}_{\mathbf{l}_i^\alpha} \to \mathbf{y}_\alpha$ denotes the cotupling $[\mathbf{y}_{s_\alpha}, [\mathbf{y}_{l_i^\alpha}]_{i \in n_\alpha}]$, for all α.*

Let $\mathbb{J}_{2,\sigma}$ denote the set of all maps $\mathscr{L}_2'(j_{2,\alpha})$ in σ-$\mathbf{Trans}_2$, where $\mathscr{L}_2' \colon \widehat{\mathbb{VT}[\mathbb{VT}][\mathbb{ET}]} \to \sigma$-$\mathbf{Trans}_2$ is left adjoint to the forgetful functor, say $\mathscr{U}_2'$, and $j_{2,\alpha} \colon \mathbf{y}_{\mathbf{s}(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(\mathbf{l}_i^\alpha)_V} \to \mathbf{y}_\alpha$ denotes the analogous cotupling $[\mathbf{y}_{s_{2,\alpha}}, [\mathbf{y}_{l_i^{2,\alpha}}]_{i \in n_\alpha}]$, for all α.

▶ **Proposition 59.** *We have* $\mathbf{FFBisim}(\sigma$-$\mathbf{Trans}) = \mathbb{J}_\sigma{}^{\pitchfork}$ *and* $\mathbf{FFBisim}(\sigma$-$\mathbf{Trans}_2) = \mathbb{J}_{2,\sigma}{}^{\pitchfork}$.

Let us now introduce border arities. A functor $F \colon \mathscr{C} \to \widehat{\mathbb{D}}$ to some presheaf category is **familial** iff there exists a functor $E \colon \mathrm{el}(F(1)) \to \mathscr{C}$ from the **category of elements** [25, §I.5] of $F(1)$, called the **exponent** of F, such that, we have a natural isomorphism

$$F(C)(d) \cong \sum_{o \in F(1)(d)} \mathscr{C}(E(d, o), C).$$

Intuitively, elements $o \in F(1)(d)$ are operations of output arity d, and $E(d, o)$ gives their input arity. Morphisms $u \colon d \to d'$ of $\mathbb{D}$ act on $F(C)$ by precomposition: for any $o' \in F(1)(d')$, we have a morphism $(d, o) \xrightarrow{u \restriction o'} (d', o')$ in $\mathrm{el}(F(1))$, where $o = F(1)(u)(o')$ – which we write $o = o' \cdot u$; and the map $F(C)(u) \colon F(C)(d') \to F(C)(d)$ sends any $(o', \varphi \colon E(d', o') \to C)$ to $(o, E(d, o) \xrightarrow{u \restriction o'} E(d', o') \xrightarrow{\varphi} C)$. This is the basis for defining border arities.

▶ **Definition 60.** *Consider a dynamic signature* Σ_1 *such that the composite* $\sigma\textbf{-Trans} \xrightarrow{\Sigma_1} \sigma\textbf{-Trans}_2 \xrightarrow{\mathcal{U}'_2} \mathbb{VT}\widehat{[\mathbb{VT}][\mathbb{ET}]}$ *is familial with exponent* E*. Let us fix* $\alpha \in \mathbb{ET}$ *and* $r \in \mathcal{U}'_2\Sigma_1(1)(\alpha)$*. For any* $k\colon A_k \to \alpha$ *among* $I_\alpha := \{s_{2,\alpha}, l_1^{2,\alpha}, \ldots, l_{n_\alpha}^{2,\alpha}\}$*, we have* $E(k \upharpoonright r)\colon E(A_k, r\cdot k) \to E(\alpha, r)$*. The* ***border arity*** $\mathbf{b}_r$ *of* r *is the cotupling* $[E(k \upharpoonright r)]_{k\in I_\alpha}\colon \sum_{k\in I_\alpha} E(A, r\cdot k) \to E(\alpha, r)$*.*

▶ **Theorem 61.** *For any dynamic signature* $\Sigma_1\colon \sigma\textbf{-Trans} \to \sigma\textbf{-Trans}_2$ *such that* $\mathcal{U}'_2\Sigma_1$ *is familial,* Σ_1 *preserves functional flexible bisimulations iff all border arities are* $\mathbb{J}_\sigma$*-cofibrations.*

Proof sketch for "if", see §B. Consider any $(u,v)\colon \mathcal{L}'_2(j_{2,\alpha}) \to \Sigma_1(f)$, with $f\colon A \to B$ in $\mathbf{FFBisim}(\sigma\textbf{-Trans})$. By adjunction, we get $(\tilde{u},\tilde{v})\colon j_{2,\alpha} \to \mathcal{U}'_2(\Sigma_1(f))$. Letting r be the composite $\mathbf{y}_\alpha \xrightarrow{\tilde{v}} \mathcal{U}'_2(\Sigma_1(B)) \xrightarrow{\mathcal{U}'_2(\Sigma_1(!))} \mathcal{U}'_2(\Sigma_1(1))$, we use familiality to factor $(\tilde{u},\tilde{v})$ as the solid part below. The result follows from finding k as shown, by $\mathbf{b}_r \in {}^{\pitchfork}(\mathbb{J}_\sigma{}^{\pitchfork})$ and $f \in \mathbb{J}_\sigma{}^{\pitchfork}$.

$$\begin{array}{ccccc}
\mathbf{y}_{\mathrm{s}(\alpha)_D} + \sum_{i\in n_\alpha} \mathbf{y}_{(\mathrm{l}_i^\alpha)_V} & \xrightarrow{\quad[\mathcal{U}'_2(\Sigma_1(in_k))\circ(r\cdot k,\mathrm{id})]_{k\in I_\alpha}\quad} & \mathcal{U}'_2(\Sigma_1(\sum_{k\in I_\alpha} E(A_k, r\cdot k))) & \xrightarrow{\mathcal{U}'_2(\Sigma_1(\psi))} & \mathcal{U}'_2(\Sigma_1(A)) \\
{\scriptstyle j_{2,\alpha}}\downarrow & & {\scriptstyle \mathcal{U}'_2(\Sigma_1(\mathbf{b}_r))}\downarrow & \overset{\mathcal{U}'_2(\Sigma_1(k))}{\dashrightarrow} & \downarrow{\scriptstyle \mathcal{U}'_2(\Sigma_1(f))} \\
\mathbf{y}_\alpha & \xrightarrow[(r,\mathrm{id})]{} & \mathcal{U}'_2(\Sigma_1(E(\alpha,r))) & \xrightarrow[\mathcal{U}'_2(\Sigma_1(\varphi))]{} & \mathcal{U}'_2(\Sigma_1(B))
\end{array}$$ ◀

▶ **Example 62.** Let us now sketch a proof of Theorem 7. By Theorems 52 and 61 and Proposition 57, it suffices to reconstruct the border arity of each rule. We only treat rule (SA) for lack of space: its border arity is the bottom morphism in

$$\begin{array}{ccc}
\mathcal{L}(0_{\mathbf{p}} + 0_{\mathbf{c}}) & \xrightarrow{\mathcal{L}[s_{[\mathbf{c}]}, l_{[\mathbf{c}]}]} & \mathcal{L}[\mathbf{c}] \\
{\scriptstyle (e_1, E[v\ \square])}\downarrow & & \downarrow \\
\mathcal{L}(0_{\mathbf{v}} + 0_{\mathbf{p}} + 0_{\mathbf{c}}) & \longrightarrow & A,
\end{array}$$

with hopefully clear notation.

▶ **Example 63.** This also works for PCF as in [17], which we omit for lack of space.

7 Conclusion and perspectives

We have introduced a categorical framework for applicative bisimilarity in the presence of operations on terms other than substitution, and of terms as labels. We have furthermore provided a notion of signature for generating instances of this framework, and proved that under suitable hypotheses, notably preservation of functional flexible bisimulations, applicative bisimilarity in the generated instance is a congruence. We have finally exhibited a more concrete sufficient condition in terms of border arities being cofibrations, which has allowed us to recover congruence of applicative bisimilarity for λ-calculus with delimited control operators and PCF.

For future work, we would be interested in further generalising the framework to cover a kind of adaptation of Howe's method that still eludes our abstraction efforts, namely (early-style) higher-order process calculi [23].

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A Proof of Theorem 52

We assume given a Howe context $\mathbb{H} = (\mathbb{VT}, \mathbb{ET}, \mathbf{s}, \mathbf{t}, \mathbf{l})$. To ease readability, we introduce some notations.

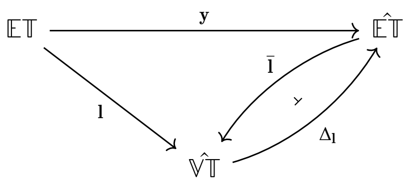
► **Notation 5.** *For any $G = (D, V, E, \gamma, \partial) \in \mathbb{H}$-$\mathbf{Trans}_2$, we let $G_{D,V}$ denote the underlying triple $(D, V, \gamma\colon D \to V) \in \widehat{\mathbb{VT}}^2$, G_0 denote V, G_1 denote E, and G_s denote D. Furthermore, following Notation 2, we denote, e.g., by $\Delta_{2,\mathbf{s},\mathbf{l}}$ the functor mapping any $\gamma\colon D \to V$ to $\Delta_{\mathbf{s}}(D) \times \Delta_{\mathbf{l}}(V)$. Finally, we sometimes treat the projection $G \mapsto G_{D,V}$ as an implicit coercion. E.g., we write $\Delta_{2,\mathbf{s},\mathbf{l}}(G)$ for $\Delta_{\mathbf{s}}(D) \times \Delta_{\mathbf{l}}(V)$.*

A.1 Basic properties of flexible bisimulation

In this section, we establish basic properties of flexible bisimulations.

► **Proposition 64.** *The functor $\Delta_{\mathbf{l}}$ is a right adjoint, hence in particular it preserves all limits.*

Proof. The functor $\Delta_{\mathbf{l}}$ is the nerve functor of $\mathbf{l}$. It is right adjoint to the left Kan extension of $\mathbf{l}$ along the Yoneda embedding, as in the following diagram.



◄

Regarding preservation of colimits, the fact that any $\mathbf{l}(c)$ is a finite coproduct of representables entails:

► **Proposition 65.** *The functor $\Delta_{\mathbf{l}}$ is algebraic, and preserves epimorphisms.*

Proof. Just for making the proof slicker, we rely on the well-known facts [2] that in presheaf categories preserving filtered colimits and reflexive coequalisers is equivalent to preserving *sifted* colimits. Furthermore, just as the covariant hom of any finitely presentable object preserves filtered colimits, in a presheaf category the covariant hom of any finite coproduct of representable objects preserves sifted colimits, hence epimorphisms. The latter fact deals with the second statement.

For the first, for any sifted colimit $\mathrm{colim}_i\, X_i$ and $c \in \widehat{\mathbb{C}}$:

$$\begin{aligned}
\Delta_{\mathbf{l}}(\mathrm{colim}_i\, X_i)(c) &= \widehat{\mathbb{VT}}(\mathbf{l}(c), \mathrm{colim}_i\, X_i) \\
&= \mathrm{colim}_i\, \widehat{\mathbb{VT}}(\mathbf{l}(c), X_i) \quad (\mathbf{l}(c) \text{ a finite coproduct of representables}) \\
&= \mathrm{colim}_i\, \Delta_{\mathbf{l}}(X_i)(c).
\end{aligned}$$

◄

► **Proposition 66.** *All functors $\Delta, \Delta_{\mathbf{l}}, \Delta_{\mathbf{s}}, \Delta_{\mathbf{t}}, \Delta_{\mathbf{s},\mathbf{l}}, \ldots$ are algebraic right adjoints (and preserve epimorphisms).*

Proof. Let us first deal with algebraicity. Because algebraic functors are closed under pointwise products, it suffices to deal with each of $\Delta_{\mathbf{l}}$, $\Delta_{\mathbf{s}}$, and $\Delta_{\mathbf{t}}$ in isolation: $\Delta_{\mathbf{s}}$ and $\Delta_{\mathbf{t}}$ are, as restriction functors; $\Delta_{\mathbf{l}}$ is by Proposition 65. Finally, in presheaf categories, being algebraic entails preservation of epimorphisms.

For right adjointness, as right adjoints are closed under pointwise products (under (co)completeness conditions satisfied here), it suffices to show that each of Δ_{l}, Δ_{s}, and Δ_{t} is a right adjoint. Again, Δ_{s} and Δ_{t} are, as restriction functors; and Δ_{l} is by Proposition 64. ◂

▸ **Proposition 67.** *All functors $\Delta_2, \Delta_{2,\mathrm{l}}, \Delta_{2,\mathrm{s}}, \Delta_{2,\mathrm{t}}, \Delta_{2,\mathrm{s},\mathrm{l}}, \ldots$ are algebraic right adjoints and preserve epimorphisms.*

Proof. Algebraic functors between presheaf categories automatically preserve epimorphisms, so it suffices to prove that all these functors are algebraic right adjoints.

Algebraic right adjoints being closed under pointwise finite products, it further suffices to prove that each of $\Delta_{2,\mathrm{l}}$, $\Delta_{2,\mathrm{s}}$, and $\Delta_{2,\mathrm{t}}$ is an algebraic right adjoint. Now each of these functors $\Delta_{2,x}$ is the corresponding functor Δ_x, precomposed with one of the projections $\widehat{\mathbb{VT}}^2 \to \widehat{\mathbb{VT}}$. But each Δ_x is an algebraic right adjoint by Proposition 66, and projections, being restriction functors, are left and right adjoints, hence algebraic right adjoints, hence the result. ◂

▸ **Lemma 68.** *In any presheaf category, for any commuting diagram of the form*

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ C' & \twoheadrightarrow & C & \longrightarrow & D \end{array}$$

if the exterior rectangle is a pointwise weak pullback and the marked morphism is epi, then so is the right-hand square.

Proof. Straightforward, using the fact that any morphism $\mathbf{y}_c \to C$ from some representable presheaf lifts to C' because epis are pointwise in presheaf categories. ◂

▸ **Proposition 69.** *A morphism $R \to X$ of diplopic $\mathbb{H}$-transition systems is a functional flexible bisimulation iff the following square is a pointwise weak pullback.*

$$\begin{array}{ccc} R_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \Delta_{\mathrm{s}}(R_s) \times \Delta_{\mathrm{l}}(R_0) & \longrightarrow & \Delta_{\mathrm{s}}(X_s) \times \Delta_{\mathrm{l}}(X_0) \end{array}$$

▸ **Lemma 70.** *For any morphisms $R \xrightarrow{f} S \xrightarrow{g} X$ in $\mathbb{H}$-$\mathbf{Trans}_2$ such that $f_{D,V} : R_{D,V} \to S_{D,V}$ is an epi, if gf is a functional flexible bisimulation, then so is g.*

Proof. We have a diagram

$$\begin{array}{ccccc} R_1 & \longrightarrow & S_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,\mathrm{s},\mathrm{l}} R_{D,V} & \longrightarrow & \Delta_{2,\mathrm{s},\mathrm{l}} S_{D,V} & \longrightarrow & \Delta_{2,\mathrm{s},\mathrm{l}} X_{D,V}\,, \end{array}$$

and want to prove that the right-hand square is a pointwise weak pullback, knowing that the outer rectangle is one: this follows readily by Lemma 68 and Proposition 67. ◂

▸ **Corollary 71.** *For any $X \in \mathbb{H}$-$\mathbf{Trans}_2$ and span morphism $f : R \to S$ in $\mathbb{H}$-$\mathbf{Trans}_2/X^2$ such that $f_{D,V}$ is an epi, if R is a (bi)simulation, then so is S.*

▸ **Proposition 72.** *The projection functors $\mathbb{H}$-$\mathbf{Trans} \to \widehat{\mathbb{VT}}$ and $\mathbb{H}$-$\mathbf{Trans}_2 \to \widehat{\mathbb{VT}}^2$ are Grothendieck fibrations.*

Proof. This follows readily from the next lemma. ◂

▸ **Lemma 73.** *For any functor $F\colon \mathbf{A} \to \mathbf{B}$ to some category $\mathbf{B}$ with pullbacks, the projection functor $p\colon \mathbf{B}/F \to \mathbf{A}$, mapping any object $b \to Fa$ to a, is a Grothendieck fibration.*

Proof. Given any object $x\colon b \to Fa$ and morphism $f\colon a' \to a$, a cartesian lifting is given by the following pullback,

$$\begin{array}{ccc} b_{|a'} & \xrightarrow{x_{\upharpoonright f}} & b \\ {\scriptstyle x_{|f}}\downarrow & \lrcorner & \downarrow{\scriptstyle x} \\ Fa' & \xrightarrow[Ff]{} & Fa \end{array}$$

cartesianness being ensured by universal property of pullback. ◂

▸ **Definition 74.** *A span $R \to X \times Y$ of diplopic $\mathbb{H}$-transition systems (resp. diplopic σ-transition systems for any enhanced syntax σ) is a* ***flexible simulation*** *if its left-hand leg $R \to X$ is a functional flexible bisimulation, and a* ***flexible bisimulation*** *when both of its legs are.*

By convention, for any $R \in \widehat{\mathbb{VT}}^2$ and $X \in \mathbb{H}\text{-}\mathbf{Trans}_2$, a span $R \to X^2_{D,V}$ is a ***flexible bisimulation*** *when the cartesian lifting $R^{\Uparrow} \to X^2$ (in the sense of Proposition 72) of X along $R \to X^2_{D,V}$ is.*

▸ **Proposition 75.** *If a span $R \to X^2$ is a flexible (bi)simulation, then so is the cartesian lifting $R^{\Uparrow}_{D,V} \to X^2$.*

Proof. By Corollary 71 applied to the span morphism $R \to R^{\Uparrow}_{D,V}$. ◂

▸ **Lemma 76.** *Consider any pullback-preserving functor $F\colon \mathbf{A} \to \mathbf{B}$ between categories with pullbacks and (strong epi-mono) factorisations. Then:*

(i) *A morphism (f,g) in $\mathbf{B}/F$ is monic iff both f and g are.*

(ii) *A morphism (f,g) in $\mathbf{B}/F$ is a strong epi iff both f and g are.*

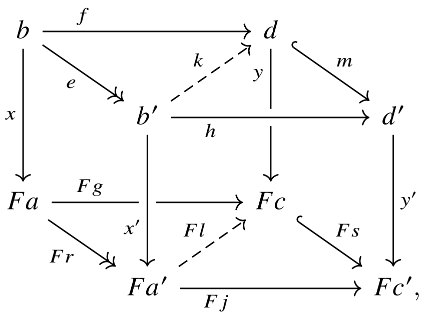
(iii) *The forgetful functor $\mathbf{B}/F \to \mathbf{B} \times \mathbf{A}$ creates, hence preserves, (strong epi-mono) factorisations.*

Proof. First of all, the forgetful functor creates all colimits, and all limits that F preserves, hence in particular pullbacks. Furthermore, in any category $\mathbf{C}$, a morphism $f\colon X \to Y$ is monic iff its **self square**

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

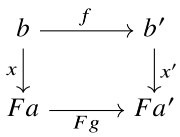
is a pullback. Thus, a morphism (f,g) in $\mathbf{B}/F$ is mono iff its self square is a pullback, iff the self squares of f and g are both pullbacks, iff f and g are both monic. This settles (i).

We next deal with the ‘if’ part of (ii), consider any diagram like the solid part in

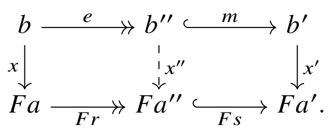


where e and r are strong epis and m and s are monos. By orthogonality, we find unique liftings k and l making all four triangles commute (without F on the bottom face). It remains to show that the vertical, diagonal square commutes: this follows by orthogonality using the fact that Fs is monic (because F preserves pullbacks, hence monos).

For (iii), consider any objects $x\colon b \to Fa$ and $x'\colon b' \to Fa'$, and let $f\colon b \to b'$ and $g\colon a \to a'$ make the following diagram commute.



Let now $b \overset{e}{\twoheadrightarrow} b'' \overset{m}{\hookrightarrow} b'$ and $a \overset{r}{\twoheadrightarrow} a'' \overset{s}{\hookrightarrow} a'$ be (strong epi-mono) factorisations of f and g, respectively. Because F preserves monos, Fs is a mono, hence by orthogonality we find a unique lifting making both squares commute in



Furthermore, by (i) and (ii), this lifting is in fact a (strong epi-mono) factorisation of (f,g), as desired. Preservation follows from (strong epi-mono) factorisations being unique up to unique isomorphism and existing in $\mathbf{B} \times \mathbf{A}$ by hypothesis.

Finally, for the ‘only if’ part of (ii): a morphism is a strong epi iff the monic part of its (strong epi-mono) factorisation is an isomorphism. So given a strong epi (e,r) in $\mathbf{B}/F$, we compute its (strong epi-mono) factorisations $m \circ e'$ and $s \circ r'$ of e and r, respectively. By (iii), they lift uniquely to a (strong epi-mono) factorisation $(m,s) \circ (e',r')$ of (e,r) in $\mathbf{B}/F$. But (e',r') is a strong epi by (ii), and so is (e,r) by hypothesis, and (m,s) is a mono between them, hence an isomorphism by Lemma 83. Thus, m and s are both isomorphisms, and hence e and r are both strong epis, as desired. ◀

▶ **Lemma 77.** *The forgetful functor*

$$\mathbb{H}\text{-}\mathbf{Trans}_2 \to \widehat{\mathbb{ET}} \times \widehat{\mathbb{VT}}^2$$

creates all colimits and limits, as well as (strong epi)-mono factorisations.

Proof. This is clear for colimits. For limits, the projection $\mathbb{H}\text{-}\mathbf{Trans}_2 \to \widehat{\mathbb{ET}} \times \widehat{\mathbb{VT}}^2$ creates all limits that the functor Δ_2 preserves (because $\mathbb{H}\text{-}\mathbf{Trans}_2$ is its lax limit), i.e., all of them by Proposition 67. For (strong epi)-mono factorisations, this follows by Lemma 76. ◀

▶ **Lemma 78.** *For any diplopic* $\mathbb{H}$*-transition system* X*, the forgetful functor*

$$\mathbb{H}\text{-}\mathbf{Trans}_2/X^2 \to \mathbb{H}\text{-}\mathbf{Trans}_2 \to \widehat{\mathbb{ET}} \times \widehat{\mathbb{VT}}^2$$

creates all colimits and connected limits.

Proof. The projection $(\mathbb{H}\text{-}\mathbf{Trans}_2)/X^2 \to \mathbb{H}\text{-}\mathbf{Trans}_2$ creates colimits and connected limits, as any projection from a slice category does. The result thus follows by Lemma 77. ◀

▶ **Lemma 79.** *Flexible bisimulations are closed under filtered colimits in* $\mathbb{H}\text{-}\mathbf{Trans}_2^2$*, i.e., in the arrow category of* $\mathbb{H}\text{-}\mathbf{Trans}_2$*.*

Proof. Let $(r_\infty : R_\infty \to X_\infty) = \mathrm{colim}_j(r_j : R_j \to X_j)$ denote the colimit of any filtered digaram of functional flexible bisimulations. By Lemma 78 and the fact that colimits are pointwise in the arrow category, we have

$$(R_\infty)_0 \cong \mathrm{colim}_j(R(j)_0) \qquad (R_\infty)_s \cong \mathrm{colim}_j(R(j)_s) \qquad (R_\infty)_1 \cong \mathrm{colim}_j(R(j)_1)$$

and

$$(X_\infty)_0 \cong \mathrm{colim}_j(X(j)_0) \qquad (X_\infty)_s \cong \mathrm{colim}_j(X(j)_s) \qquad (X_\infty)_1 \cong \mathrm{colim}_j(X(j)_1).$$

Furthermore, all morphisms

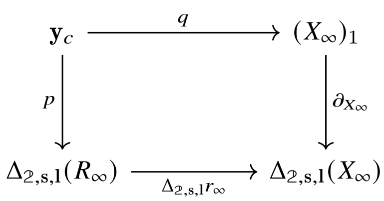
$$\gamma_{R_\infty} : (R_\infty)_s \to (R_\infty)_0 \qquad \partial_{R_\infty} : (R_\infty)_1 \to \Delta_2(R_\infty)$$

$$\gamma_{X_\infty} : (X_\infty)_s \to (X_\infty)_0 \qquad \partial_{X_\infty} : (X_\infty)_1 \to \Delta_2(X_\infty)$$

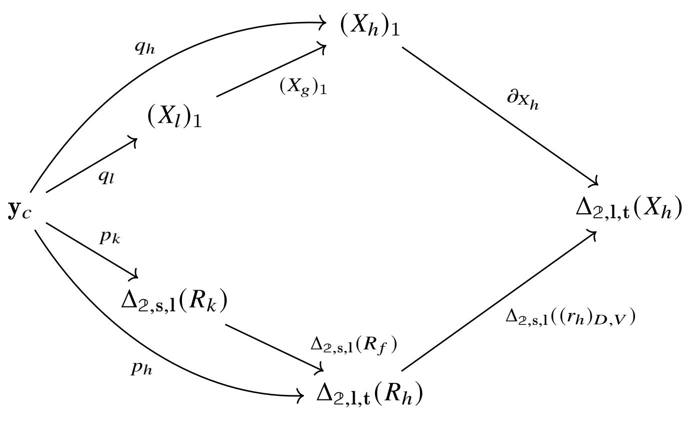
$$(r_\infty)_1 : (R_\infty)_1 \to (X_\infty)_1 \qquad (r_\infty)_s : (R_\infty)_s \to (X_\infty)_s \qquad (r_\infty)_0 : (R_\infty)_0 \to (X_\infty)_0$$

are induced by universal property.

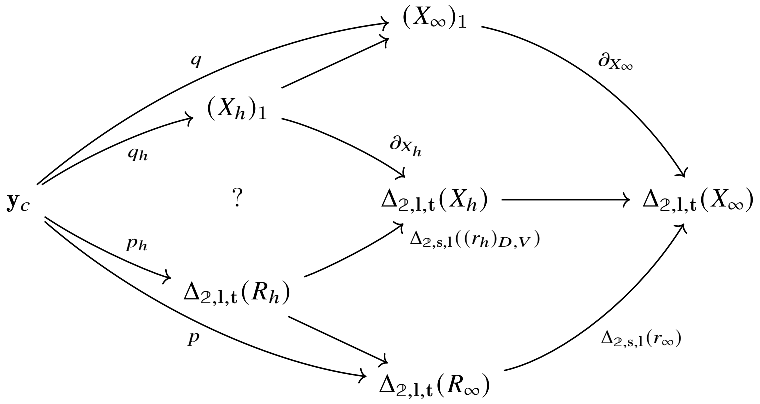
Now consider any p and q making the following diagram commute.



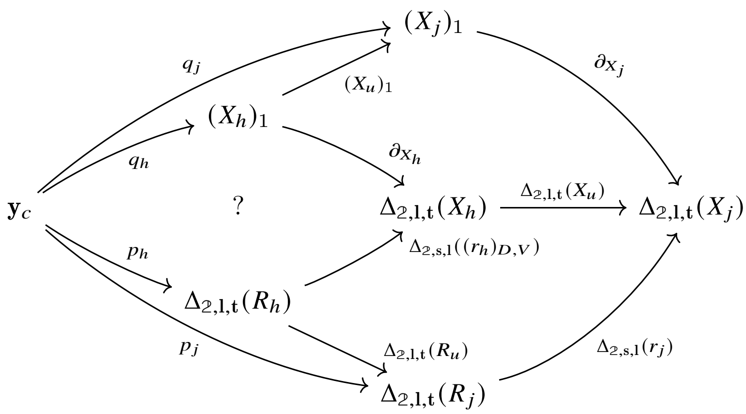
The functor $\Delta_{2,\mathrm{s},1}$ is finitary, and the object $\mathbf{y}_c$ finitely presentable, so p factors through some $\Delta_{2,\mathrm{s},1}(R_k)$, say as p_k, and q factors through some $(X_l)_1$, say as q_l. Furthermore, by filteredness, we find h and morphisms $k \xrightarrow{f} h \xleftarrow{g} l$, so that we may define p_h and q_h as in the following diagram.



Because the following diagram commutes,

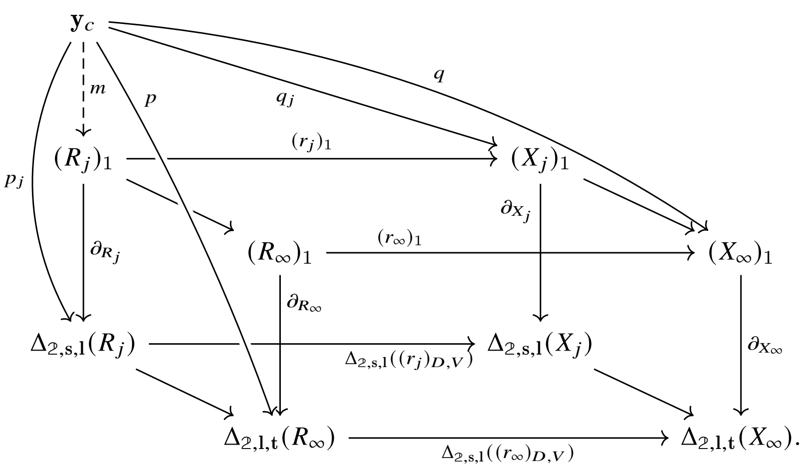


by filteredness, we find some j and morphism $u\colon h \to j$ such that $\Delta_{2,\mathrm{s},\mathrm{l}}X_u$ coequalises the question marked parallel pair above. We then define p_j and q_j by composition to obtain a commuting diagram as the following



(where again the question marked parallel pair may not commute but the exterior does).

We thus obtain a situation like



But $r_j\colon R_j \to X_j$ is a functional flexible bisimulation, so we find a mediating arrow m as shown. The composite

$$\mathbf{y}_c \xrightarrow{m} (R_j)_1 \to (R_\infty)_1$$

finally provides the desired mediating arrow. ◀

▶ **Corollary 80.** *For any diplopic $\mathbb{H}$-transition system X, flexible bisimulations $R \to X^2$ over X are closed under filtered colimits in $\mathbb{H}$-$\mathbf{Trans}_2/X^2$.*

▶ **Lemma 81.** *For any diplopic $\mathbb{H}$-transition system X, flexible bisimulations $R \to X^2$ over X are closed under span composition.*

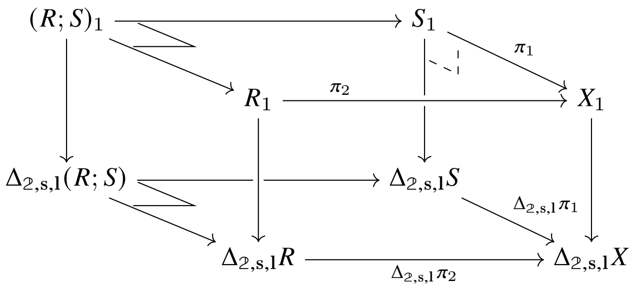
Proof. We need to show that the square

$$\begin{array}{ccc} (R;S)_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{2,s,1}(R;S) & \xrightarrow[\Delta_{2,s,1}\pi_1]{} & \Delta_{2,s,1}X \end{array}$$

is a pointwise weak pullback. By construction, this square factors as

$$\begin{array}{ccccc} (R;S)_1 & \xrightarrow{\pi_1} & R_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,s,1}(R;S) & \xrightarrow[\Delta_{2,s,1}\pi_1]{} & \Delta_{2,s,1}R & \xrightarrow[\Delta_{2,s,1}\pi_1]{} & \Delta_{2,s,1}X \end{array}$$

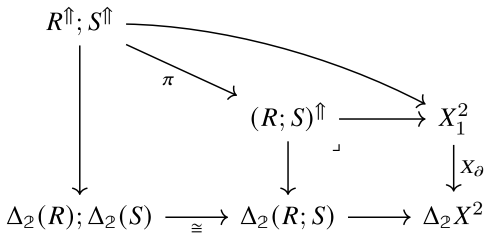
where the right-hand square is a pointwise weak pullback by hypothesis. Now the left-hand square is the left-hand face in the following diagram,



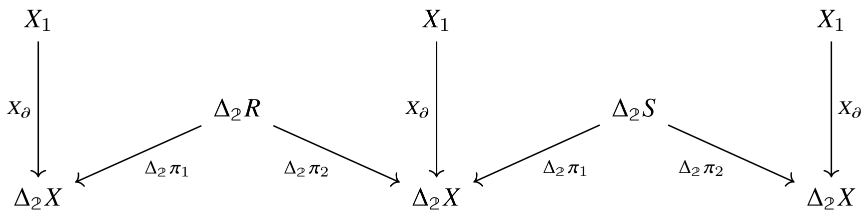
whose top and bottom faces are pullbacks by Lemma 78 and the fact that $\Delta_{2,s,1} \colon \widehat{\mathbb{VT}}^2 \to \widehat{\mathbb{ET}}$, being a right adjoint, is continuous. Since the right-hand face is a weak pullback by hypothesis, so is the left face by [18, Lemma 9.26, (i), then (ii)]. We finally conclude by [18, Lemma 9.26, (i)]. ◀

▶ **Lemma 82.** *For any diplopic $\mathbb{H}$-transition system X, flexible bisimulations $R \to X^2_{D,V}$ over X are closed under span composition.*

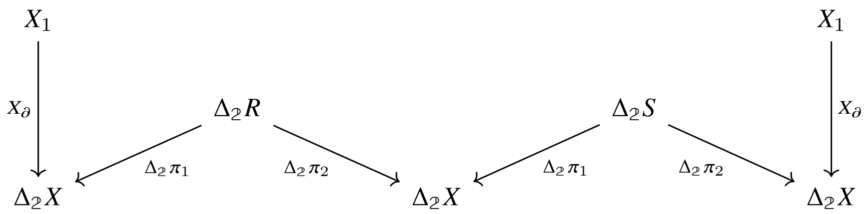
Proof. Given any two flexible bisimulations, say R and S, we observe that there is a projection $\pi \colon R^{\Uparrow}; S^{\Uparrow} \to (R;S)^{\Uparrow}$ making the following diagram commute.



To see this, we observe that by interchange of limits $R^{\Uparrow}; S^{\Uparrow}$ is the limit of



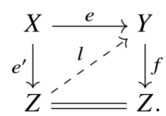
while $(R;S)^{\Uparrow}$ is the limit of the following subdiagram.



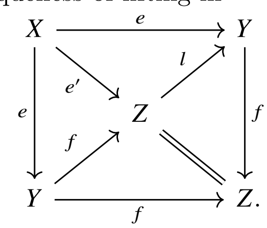
Finally, by Lemma 81, we know that $R^{\Uparrow}; S^{\Uparrow}$ is a flexible bisimulation, hence so is $R;S$ by Lemma 68. ◄

► **Lemma 83.** *For any strong epis $e\colon X \to Y$ and $e'\colon X \to Z$, any mono $f\colon Y \to Z$ such that $f \circ e = e'$ is an isomorphism.*

Proof. We find a section of f by lifting as in



But l is in fact an inverse by uniqueness of lifting in

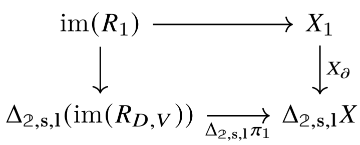


◄

► **Lemma 84.** *For any diplopic $\mathbb{H}$-transition system X, flexible bisimulations $R \to X^2$ over X are closed under images.*

Proof. Both $\widehat{\mathbb{ET}}$ and $\widehat{\mathbb{VT}}^2$ are (isomorphic to) presheaf categories, hence images are computed as (strong epi-mono) factorisations. Furthermore, Δ_2 preserves pullbacks by Proposition 67, hence by Lemma 76 the forgetful functor $\mathbb{H}\text{-}\mathbf{Trans}_2 \to \widehat{\mathbb{ET}} \times \widehat{\mathbb{VT}}^2$ creates (strong epi-mono) factorisations, hence images.

Now, consider any flexible bisimulation $p\colon R \to X^2$. As we just saw, we obtain a (strong epi-mono) factorisation of p by factoring p_1 and $p_{D,V}$. We then need to show that the square



is a pointwise weak pullback. But by Proposition 67, $\Delta_{2,\mathrm{l},\mathrm{s}}$ preserves epimorphisms. Thus, since the exterior of

$$\begin{array}{ccccc} R_1 & \longrightarrow & \mathrm{im}(R_1) & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle X_\partial} \\ \Delta_{2,\mathrm{s},\mathrm{l}}(R_{D,V}) & \twoheadrightarrow & \Delta_{2,\mathrm{s},\mathrm{l}}(\mathrm{im}(R_{D,V})) & \xrightarrow[\Delta_{2,\mathrm{s},\mathrm{l}}\pi_1]{} & \Delta_{2,\mathrm{s},\mathrm{l}}X \end{array}$$

is a pointwise weak pullback by hypothesis, we conclude by Lemma 68. ◀

A.2 Composition of flexible and rigid simulations

Our goal in this subsection is to prove the following.

▶ **Lemma 85.** *For any $\mathbb{H}$-transition system X, diplopic flexible simulation $R \to X^2$, and simulation $S_0 \to X_0^2$, equipped with a span morphism $\rho\colon R_0; S_0 \to R_0$, the relation* $\mathrm{im}(R;\theta S_0)$ *is a flexible simulation, hence so is* $\mathrm{im}(R_{D,V}; S_0)^{\Uparrow}$.

In order to prove this smoothly, we introduce the following notion of triplopic transition system.

▶ **Definition 86.** *Let $\mathbb{H}$-$\mathbf{Trans}_3$ denote the lax limit of $\widehat{\mathbb{VT}}^3 \xrightarrow{\Delta_\mathrm{s}\times\Delta_\mathrm{l}\times\Delta_\mathrm{t}} \widehat{\mathbb{ET}}$. Objects of $\mathbb{H}$-$\mathbf{Trans}_3$ are called* ***triplopic transition systems****.*

▶ **Notation 6.** *We denote by Δ_3, $\Delta_{3,\mathrm{s}}$, $\Delta_{3,\mathrm{s},\mathrm{l}}$,... the functors analogous to Δ_2, $\Delta_{2,\mathrm{s}}$, $\Delta_{2,\mathrm{s},\mathrm{l}}$,..., and often treat the projection $\mathbb{H}$-$\mathbf{Trans}_3 \to \widehat{\mathbb{VT}}^3$ as an implicit coercion, thus writing, e.g., $\Delta_{3,\mathrm{s},\mathrm{l}}X$ for any $X \in \mathbb{H}$-$\mathbf{Trans}_3$, meaning $\Delta_\mathrm{s}(X_s) \times \Delta_\mathrm{l}(X_l)$.*

A triplopic transition system X thus consists of presheaves $X_s, X_l, X_t \in \widehat{\mathbb{VT}}$ and $X_1 \in \widehat{\mathbb{ET}}$, together with a morphism $X_1 \to \Delta_\mathrm{s}(X_s) \times \Delta_\mathrm{l}(X_l) \times \Delta_\mathrm{t}(X_t)$.

▶ Remark 87. We use a boldface $\mathbb{2}$ in $\mathbb{H}$-$\mathbf{Trans}_{\mathbb{2}}$ and a normal 3 in $\mathbb{H}$-$\mathbf{Trans}_3$, to reflect the fact that any diplopic transition system $X \in \mathbb{H}$-$\mathbf{Trans}_{\mathbb{2}}$ comes with a morphism $X_s \to X_0$, while there is no such requirement for triplopic transition systems.

Let us readily notice the following useful facts.

▶ **Proposition 88.** *All functors $\Delta_3, \Delta_{3,\mathrm{l}}, \Delta_{3,\mathrm{s}}, \Delta_{3,\mathrm{t}}, \Delta_{3,\mathrm{s},\mathrm{l}}, \ldots$ are algebraic right adjoints and preserve epimorphisms.*

Proof. Algebraic functors between presheaf categories automatically preserve epimorphisms, so it suffices to prove that all these functors are algebraic right adjoints.

Algebraic right adjoints being closed under pointwise finite products, it further suffices to prove that each of $\Delta_{3,\mathrm{l}}$, $\Delta_{3,\mathrm{s}}$, and $\Delta_{3,\mathrm{t}}$ is an algebraic right adjoint. Now each of these functors $\Delta_{3,x}$ is the corresponding functor Δ_x, precomposed with one of the projections $\widehat{\mathbb{VT}}^3 \to \widehat{\mathbb{VT}}$. But each Δ_x is an algebraic right adjoint by Proposition 66, and projections, being restriction functors, are left and right adjoints, hence algebraic right adjoints, hence the result. ◀

▶ **Lemma 89.** *The forgetful functor*

$$\mathbb{H}\text{-}\mathbf{Trans}_3 \to \widehat{\mathbb{ET}} \times \widehat{\mathbb{VT}}^2$$

creates all colimits and limits, as well as (strong epi)-mono factorisations.

Proof. Just as Lemma 77. ◀

The idea of triplopic transition systems is to unify flexible and rigid bisimulation into a single framework, while allowing maximal flexibility in the choice of input and output states, and labels. Let us now define (bi)simulation in triplopic transition systems. We will then describe embeddings of transition systems and diplopic transition systems into triplopic transition systems, proving in each case that the embedding preserves and reflects bisimulation.

▶ **Definition 90.** *A morphism $f: R \to X$ of triplopic transition systems is a* ***functional bisimulation*** *iff the square*

$$\begin{array}{ccc} R_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \Delta_{3,\mathrm{s},\mathrm{l}} R & \longrightarrow & \Delta_{3,\mathrm{s},\mathrm{l}} X \end{array}$$

is a pointwise weak pullback. Spans and relations in $\mathbb{H}$-$\mathbf{Trans}_3$ *are called simulations and bisimulations analogously to the case of* $\mathbb{H}$-$\mathbf{Trans}_2$*.*

▶ **Proposition 91.** *Mapping any diplopic transition system*

$$(X_1, \gamma: X_s \to X_0, \partial: X_1 \to \Delta_{\mathrm{s}}(X_s) \times \Delta_{\mathrm{l,t}}(X_0))$$

to

$$(X_1, X_s, X_0, X_0, \partial: X_1 \to \Delta_{\mathrm{s}}(X_s) \times \Delta_{\mathrm{l,t}}(X_0))$$

yields an embedding $\iota: \mathbb{H}$-$\mathbf{Trans}_2 \to \mathbb{H}$-$\mathbf{Trans}_3$*.*

Proof. Straightforward. ◀

▶ **Notation 7.** *By composition with* $\mathbb{H}$-$\mathbf{Trans} \hookrightarrow \mathbb{H}$-$\mathbf{Trans}_2$*, we obtain a further embedding* $\mathbb{H}$-$\mathbf{Trans} \hookrightarrow \mathbb{H}$-$\mathbf{Trans}_3$*. Treating the former as an implicit coercion, we thus often also merely denote the composite by* ι*.*

▶ **Proposition 92.** *A morphism (resp. a span) of diplopic transition systems is a functional bisimulation (resp. a simulation or bisimulation) iff its embedding into triplopic transition systems is.*

Proof. Straightforward. ◀

Beyond the embedding $\mathbb{H}$-$\mathbf{Trans} \hookrightarrow \mathbb{H}$-$\mathbf{Trans}_3$ that we saw above, there is the following embedding of spans:

▶ **Proposition 93.** *For any* $X \in \mathbb{H}$-$\mathbf{Trans}$*, mapping any span* $R_0 \to X_0^2$ *in* $\widehat{\mathbb{V}\mathbb{T}}$ *to the triplopic transition system* $\theta(R_0)$ *given by* (R_0, X_0, R_0) *and* $\theta(R_0)_1 = R_0^{\uparrow}$*, i.e., given by the pullback*

$$\begin{array}{ccc} R_0^{\uparrow} & \longrightarrow & X_1^2 \\ \downarrow & \lrcorner & \downarrow \\ \Delta_{\mathrm{s}}(R_0) \times \Delta_{\mathrm{l}}(X_0) \times \Delta_{\mathrm{t}}(R_0) & \longrightarrow & \Delta X_0^2, \end{array}$$

extends to an embedding $\theta: \widehat{\mathbb{V}\mathbb{T}}/X_0^2 \to \mathbb{H}$-$\mathbf{Trans}_3/X^2$*, which we call the* ***thin*** *embedding.*

▶ Remark 94. Thinness here refers to labels, which are forced to agree on both sides of any transition in $\theta(R_0)$.

The thin embedding enables the following characterisation of bisimulation in $\mathbb{H}$-transition systems in terms of bisimulation in triplopic $\mathbb{H}$-transition systems:

▶ **Proposition 95.** *For any $X \in \mathbb{H}$-**Trans**, a span $R_0 \to X_0^2$ is a simulation (resp. bisimulation) iff $\theta(R_0) \to X^2$ is one.*

Proof. Both statements mean that the square

$$\begin{array}{ccc} R_0^{\uparrow} & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{\mathrm{s}}(R_0) \times \Delta_{\mathrm{l}}(X_0) & \xrightarrow[\pi_1]{} & \Delta_{\mathrm{s},\mathrm{l}} X_0 \end{array}$$

is a pointwise weak pullback. ◀

Finally, we have the easy

▶ **Proposition 96.** *(Bi)simulations are closed under span composition in $\mathbb{H}$-**Trans**$_3$.*

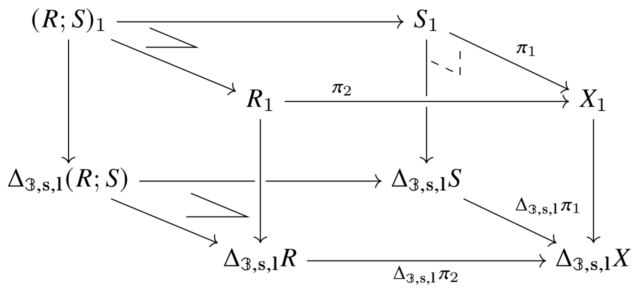
Proof. By symmetry it suffices to show that simulations are closed under span composition. Let us thus consider any simulations R and S over some $X \in \mathbb{H}$-**Trans**$_3$. We must show that the square

$$\begin{array}{ccc} (R;S)_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{3,\mathrm{s},\mathrm{l}}(R;S) & \xrightarrow[\Delta_{3,\mathrm{s},\mathrm{l}}\pi_1]{} & \Delta_{3,\mathrm{s},\mathrm{l}} X \end{array}$$

is a pointwise weak pullback. This square factors as

$$\begin{array}{ccccc} (R;S)_1 & \xrightarrow{\pi_1} & R_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{3,\mathrm{s},\mathrm{l}}(R;S) & \xrightarrow[\Delta_{3,\mathrm{s},\mathrm{l}}\pi_1]{} & \Delta_{3,\mathrm{s},\mathrm{l}}(R) & \xrightarrow[\Delta_{3,\mathrm{s},\mathrm{l}}\pi_1]{} & \Delta_{3,\mathrm{s},\mathrm{l}} X, \end{array}$$

where the right-hand square is a pointwise weak pullback by hypothesis, and the left-hand square is the left-hand face in



whose top and bottom faces are pullbacks by Lemma 78 and the fact that $\Delta_{3,\mathrm{s},\mathrm{l}}$, being a right adjoint, is continuous. Since the right-hand face is a pointwise weak pullback by hypothesis, so is the left-hand face by [18, Lemma 9.26, (i), then (ii)]. The whole rectangle thus is a pointwise weak pullback by [18, Lemma 9.26, (i)], as desired. ◀

▶ **Proposition 97.** *Triplopic (bi)simulations are closed under images.*

Proof. By symmetry it suffices to treat the case of simulations. Let $R \to X^2$ be any triplopic simulation. Then by Proposition 89 we need to prove that the right-hand square below is a pointwise weak pullback,

$$
\begin{array}{ccccc}
R_1 & \twoheadrightarrow & \mathrm{im}(R_1) & \hookrightarrow & X_1^2 \\
\downarrow & & \downarrow & & \downarrow \\
\Delta_{3,\mathrm{s},1}R & \twoheadrightarrow & \Delta_{3,\mathrm{s},1}\,\mathrm{im}(R) & \hookrightarrow & \Delta_{3,\mathrm{s},1}X^2
\end{array}
$$

which is the case by Lemma 68 and the fact that $\Delta_{3,\mathrm{s},1}$ preserves epis by algebraicity (Lemma 88). ◂

▸ **Lemma 98.** *Given a retraction $R \twoheadrightarrow S$ over any X^2 in $\mathbb{H}$-**Trans**$_3$, if S is a simulation, then so is $\mathrm{im}(R)$.*

Proof. The given retraction and its section yield morphisms

$$\mathrm{im}(R) \to \mathrm{im}(S) \qquad \text{and} \qquad \mathrm{im}(S) \to \mathrm{im}(R),$$

hence $\mathrm{im}(R) \cong \mathrm{im}(S)$, so we conclude by Lemma 97. ◂

Proof of Lemma 85. The morphism $\tilde{\rho}\colon R;\iota S \to R;\theta S$ defined by the triple

$$\mathrm{id}\colon R_0;S_0 \to R_0;S_0 \qquad \rho\colon R_0;S_0 \to R_0 \qquad \mathrm{id}\colon R_0;S_0 \to R_0;S_0$$

admits a section, namely the morphism $R;\theta S \to R;\iota S$ defined by

$$\mathrm{id}\colon R_0;S_0 \to R_0;S_0 \qquad R_0 \cong R_0;X_0 \to R_0;S_0 \qquad \mathrm{id}\colon R_0;S_0 \to R_0;S_0$$

(induced by reflexivity of S_0). Thus, $\mathrm{im}(R;\iota S)$ is a triplopic simulation by Lemma 98, hence a diplopic one by Proposition 92. Finally, $\mathrm{im}(R_{D,V};S_0)^{\Uparrow}$ is a flexible simulation by Proposition 75. ◂

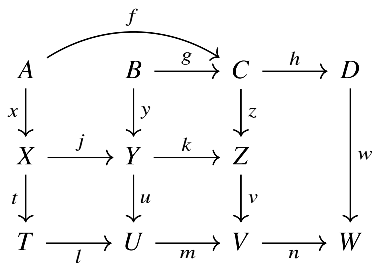
A.3 Fundamental property of flexible bisimulation

In this section, we reduce the theorem to a certain result involving flexible bisimulations, using the following fundamental property of flexible bisimulation:

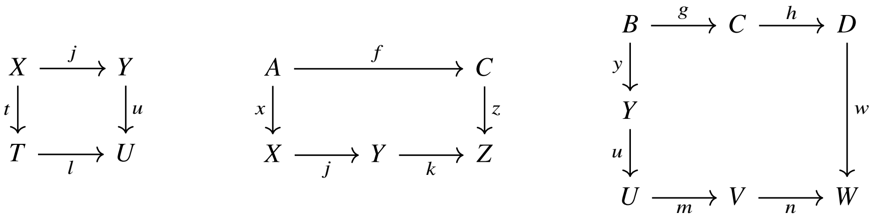
▸ **Proposition 99.** *For any $X \in \mathbb{H}$-**Trans** and reflexive, flexible bisimulation $R \to X^2$, $R_0 \to X_0^2$ is a bisimulation.*

We need the following lemma.

▸ **Lemma 100.** *Consider any commuting diagram of the following form*



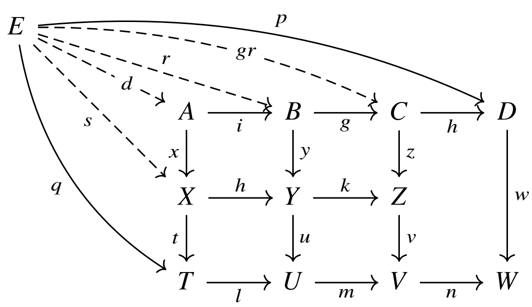
(i.e., all three squares and the rectangle commute, plus $zf = kjx$), such that all three squares below are weak pullbacks.



Then, the exterior is again a weak pullback.

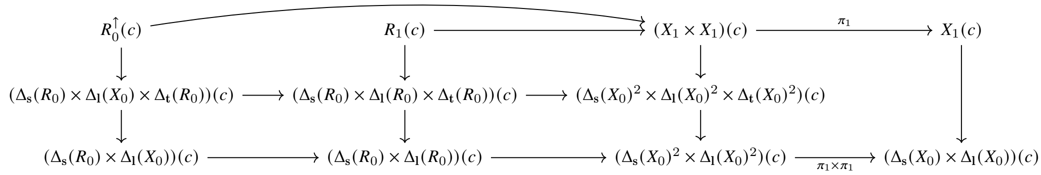
Proof. First, we find $i\colon A \to B$ such that $gi = f$ and $uyi = ltx$, by weak universal property of B.

Now, consider any cone (p, q) as shown below.



By weak universal property of B, we find $r\colon E \to B$ such that $hgr = p$ and $vyr = lq$. By weak universal property of X, we then find a morphism $s\colon E \to X$ such that $us = q$ and $hs = yr$. Finally, by weak universal property of A, we find the desired morphism $d\colon E \to A$ such that $xd = s$ and $gid = gr$. Please note that nothing here guarantees that $id = r$, nor that $yi = hx$, but this does invalidate the result. ◀

Proof of Proposition 99. By symmetry, it suffices to check that the first projection $\pi_1\colon R_0 \to X_0$ is a simulation. For any $c \in \mathbb{ET}$, we form the following diagram,



and conclude by Lemma 100. To check that it applies, we observe that

- the first requirement holds easily (the bottom left square is easily seen to be a pullback);
- the second requirement holds by construction of $R_0^{\Uparrow}$; and
- the last requirement holds by hypothesis that R is a flexible bisimulation. ◀

Let us now use the fundamental property (Proposition 99) of flexible bisimulation to reduce congruence of bisimilarity to the search for a suitable flexible enhanced bisimulation.

▶ **Corollary 101.** *Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i\in n})$. Let σ denote the generated enhanced syntax $\sigma(\mathbf{d})$. Let X be any σ-transition system, and suppose that there exists a reflexive, enhanced, flexible bisimulation relation $R \to X^2$ such that $\sim_X^\sigma \subseteq R_0$ and R_0 is a congruence. Then enhanced bisimilarity $\sim_X^\sigma$ is a congruence.*

Proof. Consider any reflexive, enhanced, flexible bisimulation relation $R \to X^2$ such that R_0 contains enhanced bisimilarity and is a congruence. By Proposition 99, R_0 is an enhanced bisimulation, so by terminality of $\sim^\sigma_X$, we have $R_0 \subseteq \sim^\sigma_X$, hence morphisms

$$\Sigma_0(\sim^\sigma_X) \to \Sigma_0(R_0) \to R_0 \to \sim^\sigma_X$$

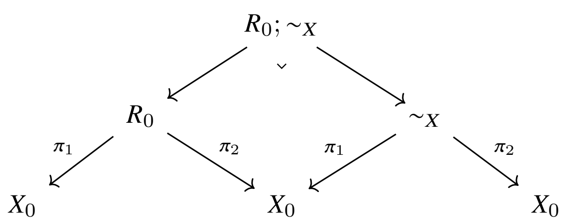
over X_0^2. ◀

A.4 Howe closure: basic properties

In this section, we introduce our candidate reflexive, enhanced, flexible bisimulation relation $R \to \mathbf{Z}^2$ such that $\sim_{\mathbf{Z}} \subseteq R_0$ and R_0 is a congruence. As is standard, we

- construct it directly as a congruence,
- prove that it is reflexive and enhanced (relatively easily), and, finally,
- struggle to prove that it (or rather its transitive closure) is a flexible bisimulation.

► **Definition 102.** *Let the* ***Howe functor*** $\Sigma_0^{+;\sim_X} : \widehat{\mathbb{VT}}/X_0^2 \to \widehat{\mathbb{VT}}/X_0^2$ *map any* $R_0 \to X_0^2$ *to the coproduct span* $\Sigma_0(R_0) + (R_0; \sim_X)$*, where the second term more concretely denotes the following composite span.*



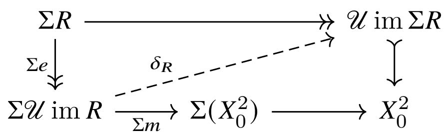
Let the ***proof-relevant Howe closure*** $R_0^{\odot}$ *be the free* $\Sigma_0^{+;\sim_X}$*-algebra on* R_0*, and the* ***(proof-irrelevant, or relational) Howe closure*** $R_0^{\bullet}$ *denote the image of* $R_0^{\odot} \to X_0^2$*.*

The Howe functor is a finitary endofunctor on a presheaf category, so we have [27]:

► **Proposition 103.** *The free* $\Sigma_0^{+;\sim_X}$*-algebra on any* R_0 *exists and is computed by the standard initial chain, and the forgetful functor* $\Sigma_0^{+;\sim_X}\text{-}\mathbf{alg} \to \widehat{\mathbb{VT}}/X_0^2$ *is finitary monadic.*

► **Proposition 104.** *Let* $\mathcal{U} : \mathbf{Sub}(X_0^2) \hookrightarrow \widehat{\mathbb{VT}}/X_0^2$ *denote the canonical embedding, and let* $\Sigma = \Sigma_0^{+;\sim_X}$ *just for this proposition. The composite endofunctor* $\mathrm{im} \circ \Sigma^* \circ \mathcal{U}$ *on* $\mathbf{Sub}(X_0^2)$ *is a monad, which is in fact the free monad on* $\mathrm{im} \circ \Sigma \circ \mathcal{U}$*. Consequently, the relational Howe closure* $(\mathcal{U}R)^{\bullet}$ *on a relation* $R \in \mathbf{Sub}(X_0^2)$ *is the free* $(\mathrm{im} \circ \Sigma \circ \mathcal{U})$*-algebra over* R*.*

Proof. Using algebraicity of Σ_0, it is straightforward to show that Σ preserves epimorphisms. For any $R \in \widehat{\mathbb{VT}}/X_0^2$, letting $T = \mathcal{U} \circ \mathrm{im}$ denote the monad induced by the adjunction $\mathrm{im} \dashv \mathcal{U}$, we thus have by unique lifting a morphism $\delta_R : \Sigma T R \to T \Sigma R$ as in the following diagram,



where $R \to X_0^2$ factors as $e \circ m$ and the last horizontal morphism is

$$\Sigma_0^{+;\sim_X}(X_0^2) = \Sigma_0(X_0^2) + (X_0^2); \sim_X \xrightarrow{[\langle a \circ \Sigma_0(\pi_1), a \circ \Sigma_0(\pi_2)\rangle, \langle \pi_1 \circ \pi_1, \pi_2 \circ \pi_2 \rangle]} X_0^2.$$

The result thus follows from the next lemma. ◀

▶ **Lemma 105.** *Consider a full, reflective embedding $U\colon \mathscr{D} \hookrightarrow \mathscr{C}$ from some poset $\mathscr{D}$ into a locally finitely presentable category $\mathscr{C}$, say with left adjoint $L\colon \mathscr{C} \to \mathscr{D}$, together with a finitary endofunctor Σ on $\mathscr{C}$. Furthermore, assume given a functor distributive law, i.e., a natural transformation $\delta\colon \Sigma T \to T\Sigma$, where $T := UL$ denotes the induced monad. Then, $L\Sigma^* U$ is the free monad on $L\Sigma U$, hence in particular the free $L\Sigma U$-algebra on any $D \in \mathscr{D}$ is $L\Sigma^* UD$.*

▶ **Lemma 106.** *In the setting of Lemma 105, all objects of the form $UD \in \mathscr{C}$ are **subterminal**, in the sense that any two parallel morphisms to UD are equal.*

Proof. Consider any $f, g\colon C \to UD$. By adjunction, these correspond bijectively to morphisms $\tilde{f}, \tilde{g}\colon LC \to D$, which, because $\mathscr{D}$ is a poset, are equal. ◀

Proof of Lemma 105. By [27], Σ admits a free monad Σ^*.

Furthermore, by Lemma 106, the given functor distributive law δ is in fact a **functor-monad distributive law**, in the sense that it commutes with the unit and multiplication of T.

Now, by a reasoning analogous to [3], functor-monad distributive laws $\delta\colon \Sigma T \to T\Sigma$ correspond bijectively to liftings of the monad T to $\Sigma\text{-}\mathbf{alg}$, i.e., monads T^δ on $\Sigma\text{-}\mathbf{alg}$ making the following square commute,

$$\begin{array}{ccc} \Sigma\text{-}\mathbf{alg} & \xrightarrow{T^\delta} & \Sigma\text{-}\mathbf{alg} \\ \downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow[T]{} & \mathscr{C} \end{array}$$

whose multiplication and unit are mapped by the forgetful functor to those of T. The given functor-monad distributive law δ thus corresponds to such a lifting. But $\Sigma\text{-}\mathbf{alg} \cong \Sigma^*\text{-}\mathbf{Alg}$ over $\mathscr{C}$, hence we get a lifting of T to $\Sigma^*\text{-}\mathbf{Alg}$, which by [3] again amounts to a monad distributive law, say $\bar{\delta}\colon \Sigma^* T \to T\Sigma^*$.

From this, using the fact that the counit is an isomorphism (which follows from full faithfulness of U), we equip the composite $L\Sigma^* U$ with monad structure:

- the unit is the composite $R \xrightarrow{(\varepsilon^T)^{-1}} LUR \xrightarrow{L\eta^{\Sigma^*}} L\Sigma^* UR$,
- the multiplication is

$$L\Sigma^* UL\Sigma^* UR = L\Sigma^* T\Sigma^* UR \xrightarrow{L\bar{\delta}} LT\Sigma^*\Sigma^* UR \xrightarrow{\varepsilon U\mu^{\Sigma^*} UR} L\Sigma^* UR,$$

- and the monad laws hold automatically since $\mathscr{D}$ is a poset.

Moreover, given any $R \in \mathscr{D}$, the following are equivalent

- Σ^*-algebra structure (in the monad sense) on UR,
- Σ^*-algebra structure (in the functor sense) on UR,
- Σ-algebra structure on UR,
- $L\Sigma^* U$-algebra structure (in the monad sense) on R,
- $L\Sigma^* U$-algebra structure (in the functor sense) on R,
- $L\Sigma U$-algebra structure on R.

Indeed,

- Σ-algebra structure $\Sigma UR \to UR$ corresponds by adjunction to $L\Sigma U$-algebra structure $L\Sigma UR \to R$;
- Σ-algebra structure $\Sigma UR \to UR$ corresponds by universal property of Σ^* to Σ^*-algebra structure $\Sigma^* UR \to UR$ in the monad sense;

- by subterminality, Σ^*-algebra structures $\Sigma^* UR \to UR$ in the monad and functor sense are equivalent;
- by adjunction again, Σ^*-algebra structure $\Sigma^* UR \to UR$ in the functor sense is equivalent to $L\Sigma^* U$-structure $L\Sigma^* UR \to R$ in the functor sense;
- and finally, because $\mathscr{D}$ is a poset, $L\Sigma^* U$-structures $L\Sigma^* UR \to R$ in the functor and monad sense are equivalent.

We thus in particular get $(L\Sigma^* U)\text{-}\mathbf{Alg} \cong (L\Sigma U)\text{-}\mathbf{alg}$ over $\mathscr{D}$, hence the result. ◀

▶ **Definition 107.** *Let $S^{+;\sim_X}$ denote the monad induced by $\Sigma_0^{+;\sim_X}$ on $\widehat{\mathbb{VT}}/X_0^2$.*

▶ **Lemma 108.** *Let R_0' be the proof-relevant (resp. proof-irrelevant) Howe closure $R_0^{\odot}$ (resp. $R_0^{\bullet}$) of (resp. a relation) R_0. It satisfies the following properties.*
(i) *R_0' is a Σ_0-algebra;*
(ii) *there exists an action $R_0'; \sim_X \to R_0'$ over X_0^2.*
Furthermore, if $X_0 = \Sigma_0^*(\emptyset)$ is the initial Σ_0-algebra, we have:
(iii) *R_0' is reflexive,*
(iv) *there exists a morphism $\sim_X \to R_0'$ over X_0^2.*

Proof. We prove the properties for the proof-relevant Howe closure – they follow easily for the proof-irrelevant one.
(i) By definition $R_0^{\odot}$ is an $\Sigma_0^{+;\sim_X}$-algebra, hence in particular a Σ_0-algebra, or more correctly an algebra for the obvious lifting of Σ_0 to $\widehat{\mathbb{VT}}/X_0^2$.
(ii) As an $\Sigma_0^{+;\sim_X}$-algebra, $R_0^{\odot}$ is an algebra for the second term functor, i.e., a morphism of the desired form $R_0^{\odot}; \sim_X \to R_0^{\odot}$.
Let us now assume that X_0 is the initial Σ_0-algebra. Then, by initiality of X_0 and (i), there is a unique Σ_0-algebra morphism $X_0 \to R_0^{\odot}$, which witnesses reflexivity.

We then use reflexivity and (ii) to construct the following composite

$$\sim_X \cong X_0; \sim_X \to R_0^{\odot}; \sim_X \to R_0^{\odot},$$

which proves the second point. ◀

A further crucial property is:

▶ **Proposition 109.** *If X_0 is an ST-algebra, then the proof-relevant Howe closure $R_0^{\odot}$ on any R_0 is an ST-algebra, and $R_0^{\odot} \to X_0^2$ is a morphism of ST-algebras. Furthermore, the relational Howe closure $R_0^{\bullet}$ is enhanced.*

In order to prove this, we need a few intermediate steps.

▶ **Definition 110.** *For any bifunctor F on a category $\mathbf{C}$ and $F\Delta$-algebra X, let $\bar{F}$ denote the lifting of F to $\mathbf{C}/X^2$, which maps any $U \to X^2$ and $V \to X^2$ to the composite*

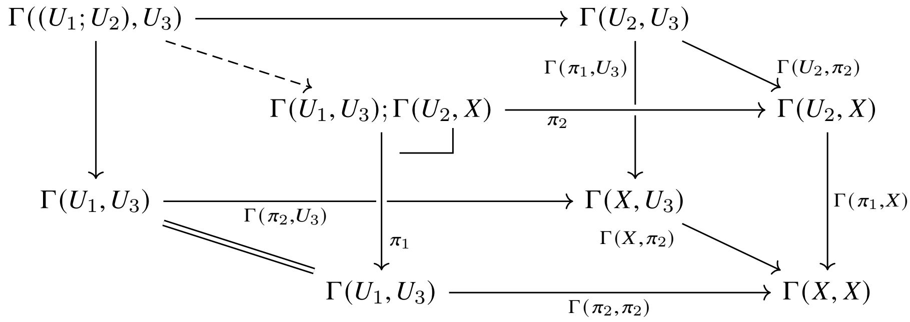
$$F(U,V) \to F(X^2, X^2) \to F(X,X)^2 \to X^2.$$

▶ **Lemma 111.** *For any bifunctor Γ on a category $\mathbf{C}$ with pullbacks, object $X \in \mathbf{C}$, and spans $u_i \colon U_i \to X^2$, for $i \in 3$, there is a morphism*

$$\Gamma((U_1; U_2), U_3) \to \Gamma(U_1, U_3); \Gamma(U_2, X)$$

of spans over X.

Proof. We construct the desired morphism by universal property of pullback, as in the following diagram.



◀

▶ **Lemma 112.** *Assume that X_0 is a σ-algebra with structure given by*

$$\mathbf{a}\colon \Sigma_0 X_0 \to X_0 \qquad \dots \qquad \mathbf{b}_i : \Gamma_i(X_0, X_0) \to X_0 \qquad \dots ,$$

and let the derived monad algebra structures be as follows.

$$\bar{\mathbf{a}}\colon SX_0 \to X_0 \qquad \dots \qquad \bar{\mathbf{b}}_{<i} : T_i X_0 \to X_0 \qquad \dots .$$

Then, for all $i \in n$, the incremental structural law

$$d_i : \Gamma_i(\Sigma_0 A, B) \to ST_i(\Gamma_i(A, ST_i B) + A + B)$$

lifts to an incremental structural law

$$\bar{d}_i : \bar{\Gamma}_i(\Sigma_0^{+;\sim_X} A, B) \to S^{+;\sim_X}\bar{T}_i(\bar{\Gamma}_i(A, S^{+;\sim_X}\bar{T}_i B) + A + B).$$

Proof. By Lemma 111, using left-cocontinuity of Γ_i, and the fact that $\sim_X$ is enhanced. ◀

Proof of Proposition 109. By Proposition 34, there exists a distributive law

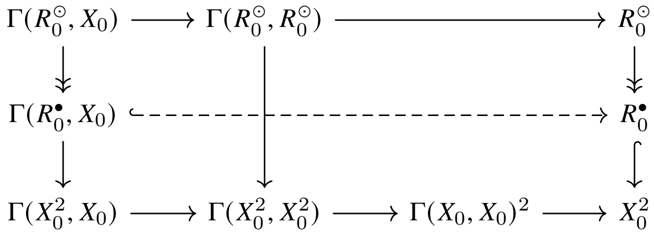
$$\bar{T}_{n+1} S^{+;\sim_X} \to S^{+;\sim_X}\bar{T}_{n+1}$$

and $\bar{T}_{n+1}$ is constant-free, hence the natural transformation $S^{+;\sim_X} \to S^{+;\sim_X}\bar{T}_{n+1}$ is an isomorphism at $\emptyset$. The proof-relevant Howe closure $R_0^{\odot} = S^{+;\sim_X}\emptyset$ thus acquires a canonical $S^{+;\sim_X}\bar{T}_{n+1}$-algebra structure. The terminal object also is one, of course, and the unique morphism to it is a $S^{+;\sim_X}\bar{T}_{n+1}$-algebra morphism, which completes the proof of the first point.

The proof-relevant Howe closure is in particular enhanced via

$$\Gamma_i(R_0^{\odot}, X) \to \Gamma_i(R_0^{\odot}, R_0^{\odot}) \to R_0^{\odot},$$

which entails enhancedness for the relational Howe closure by the fact that each Γ_i, being left-cocontinuous, preserves epimorphisms in its first argument, and that all epimorphisms are strong in presheaf categories. Indeed, we find the desired morphism by lifting as in the following diagram.



◀

A final basic property is about symmetry of the relational transitive closure of the relational Howe closure on the syntactic transition system (Proposition 116 below).

▶ **Definition 113** ([18, Definition 9.5]). *The **relational transitive closure** $R_0^{\bar{+}}$ of a span $R_0 \to X_0^2$ is the union $\bigcup_{n>0} \mathrm{im}(R_0^{;n})$, where $(-)^{;n}$ denotes iterated self-composition of spans.*

▶ **Proposition 114.** *For any span $R_0 \to X_0^2$, the relational transitive closure $R_0^{\bar{+}}$ is equipped with an action $R_0; R_0^{\bar{+}} \to R_0^{\bar{+}}$ over X_0^2.*

The proof relies on the following lemma.

▶ **Lemma 115.** *In any complete, cocomplete, regular, and locally cartesian closed category, hence in particular in any presheaf category,*
(i) *span composition preserves all colimits, on both sides, and*
(ii) *sequential composition of relations preserves all unions, on both sides.*

Proof. The pullback functor (along the relevant projection), being a left adjoint, is cocontinuous, which directly entails the first point. For the second point, in a regular category, the pullback functor preserves regular epis and monos, hence image factorisations. ◀

Proof of Proposition 114. We have

$$R_0; \bigcup_{n>0} \mathrm{im}(R_0^{;n}) \twoheadrightarrow \mathrm{im}(R_0); \bigcup_{n>0} \mathrm{im}(R_0^{;n}) \cong \bigcup_{n>1} \mathrm{im}(R_0^{;n}) \hookrightarrow \bigcup_{n>0} \mathrm{im}(R_0^{;n}),$$

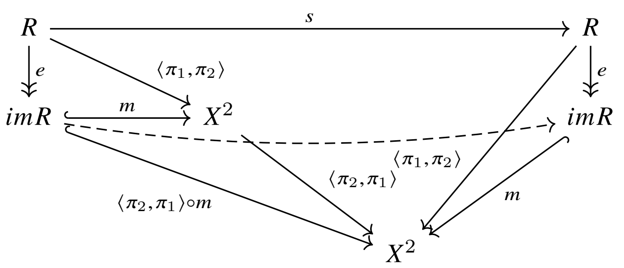
where the isomorphism holds by Lemma 115(ii). ◀

▶ **Proposition 116.** *Let again $X_0 = \Sigma_0^*(\emptyset)$. Then the relational transitive closure $\emptyset^{\bullet\bar{+}}$ of the proof-irrelevant Howe closure of $\emptyset$ is symmetric.*

▶ **Lemma 117** ([18, Lemma 9.10]). *For any span $R_0 \to \mathbf{Z}_0^2$, if there exists a span morphism $R_0 \to R_0^{\bar{+}\dagger}$, then $R_0^{\bar{+}}$ is symmetric.*

▶ **Lemma 118.** *If a span R is symmetric, in the sense that there is a morphism $R^\dagger \to R$ over X_0^2, then so is its induced relation.*

Proof. We proceed as in the following diagram.



◀

Proof of Proposition 116. By the lemma, it suffices to construct a morphism $\emptyset^\bullet \to \emptyset^{\bullet\bar{+}\dagger}$. Thus, by Proposition 104, it suffices to endow $\emptyset^{\bullet\bar{+}\dagger}$ with algebra structure for the endofunctor $S \mapsto \mathrm{im}(\Sigma_0^{+;\sim X}(S))$ on $\mathbf{Sub}(X_0^2)$. For this, because $\Sigma_0^{+;\sim X}$ is algebraic, it suffices to endow $\emptyset^{\bullet\bar{+}\dagger}$ with $\Sigma_0^{+;\sim X}$-algebra structure.

We first equip it with $(-;\sim_X)$-algebra structure. We need to find a morphism $\emptyset^{\bullet\mp\dagger};\sim_X \to \emptyset^{\bullet\mp\dagger}$ over X_0^2, or equivalently by applying the involution $(-)^\dagger$, a morphism ${\sim_X}^\dagger;\emptyset^{\bullet\mp} \to \emptyset^{\bullet\mp}$. We pick the composite

$$ {\sim_X}^\dagger;\emptyset^{\bullet\mp} \to \sim_X;\emptyset^{\bullet\mp} \to \emptyset^{\bullet};\emptyset^{\bullet\mp} \to \emptyset^{\bullet\mp}, $$

where

- the first morphism is symmetry of $\sim_X$,
- the second morphism is that of Lemma 108,
- the last morphism is the action from Proposition 114.

This leaves us with the task of equipping $\emptyset^{\bullet\mp\dagger}$ with algebra structure for the lifting of Σ_0 to $\mathbf{Sub}(X_0^2)$, for which it suffices, by algebraicity of Σ_0, to equip it with algebra structure for the lifting of Σ_0 to $\widehat{\mathbb{VT}}/X_0^2$, say $\bar{\Sigma}_0$. By [18, Corollary 9.8], we have $\emptyset^{\bullet\mp\dagger} \cong \emptyset^{\bullet\dagger\mp}$, and by [18, Lemma 9.9], $\emptyset^{\bullet\dagger\mp}$ is the colimit of the chain

$$ X_0 \to \mathrm{im}(\emptyset^{\bullet\dagger}) \cong \mathrm{im}(\emptyset^{\bullet\dagger};X_0) \to \mathrm{im}(\emptyset^{\bullet\dagger};\emptyset^{\bullet\dagger}) \cong \mathrm{im}(\emptyset^{\bullet\dagger};\emptyset^{\bullet\dagger};X_0) \to \mathrm{im}(\emptyset^{\bullet\dagger};\emptyset^{\bullet\dagger};\emptyset^{\bullet\dagger}) \to \dots $$

in $\widehat{\mathbb{VT}}/X_0^2$. But $\bar{\Sigma}_0$ is algebraic, hence the forgetful functor $\bar{\Sigma}_0\text{-}\mathbf{alg} \to \widehat{\mathbb{VT}}/X_0^2$ creates filtered colimits, hence in particular colimits of chains. It thus suffices to lift the above chain to $\bar{\Sigma}_0\text{-}\mathbf{alg}$. Furthermore, because all objects of the chain are relations, they are subterminal, hence all morphisms will automatically lift to $\bar{\Sigma}_0\text{-}\mathbf{alg}$ if the objects do. Finally, the forgetful functor $\bar{\Sigma}_0\text{-}\mathbf{alg} \to \widehat{\mathbb{VT}}/X_0^2$ creates limits, and $\emptyset^\bullet$ possesses Σ_0-algebra structure by Lemma 108, hence so does $\emptyset^{\bullet\dagger}$. ◀

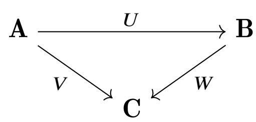
To conclude this section, we use the basic facts we just proved to reduce the main result to the fact that $\emptyset^\bullet$ is a flexible simulation.

▶ **Proposition 119.** *Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i\in n})$, and suppose that $\emptyset^\bullet_{\mathbf{Z}}$ is a flexible simulation. Then enhanced bisimilarity on $\mathbf{Z}$ is a congruence.*

▶ Remark 120. Let us recall that by Definition 74, $R_0 \to X_0^2$ in $\widehat{\mathbb{VT}}$ is a flexible simulation when its cartesian lifting $R_0^{\Uparrow} \to X^2$ is.

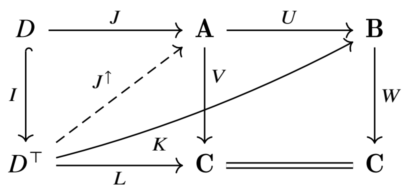
We will rely on the following lemmas.

▶ **Lemma 121.** *Consider any commutative diagram of functors between locally small categories*



If V and W create colimits of a certain shape D, and W preserves them (typically if $\mathbf{C}$ has them), then U creates them.

Proof. Consider any functor $J\colon D \to \mathbf{A}$ and colimiting cocone $K\colon D^\top \to \mathbf{B}$ for $U \circ J$. Because W preserves colimits of shape D, $W \circ K$ is colimiting for $W \circ U \circ J$, hence because V creates colimits, we find a unique lifting $J^\uparrow$ such that $J^\uparrow \circ I = J$ and $V \circ J^\uparrow = L := W \circ K$, as in the following diagram.



But now $U \circ J^{\uparrow}$ and K both are candidate liftings for the outer rectangle, so by uniqueness in the creation of colimits by W they are equal, and thus $J^{\uparrow}$ is a lifting for the original square (J, K).

Furthermore, any lifting for (J, K) induces one for (J, L), hence should be equal to $J^{\uparrow}$, which proves uniqueness.

Finally, $J^{\uparrow}$ is colimiting because V creates colimits of shape D. ◀

▶ **Definition 122.** *Given a bifunctor* $\Gamma\colon \mathbf{C}^2 \to \mathbf{C}$ *and an object* X*, a* (Γ, X)***-premodule*** *is an object* M *equipped with an* ***action****, i.e., a morphism* $r\colon \Gamma(M, X) \to M$*. A morphism of* (Γ, X)*-premodules is a morphism commuting with action. We let* $(\Gamma, X)\text{-}\mathbf{Mod}$ *denote the category of* (Γ, X)*-premodules.*

▶ **Terminology 1.** *When* Γ *is clear from context, we often omit it and talk about* X*-premodules and* $X\text{-}\mathbf{Mod}$*.*

▶ Remark 123. An enhanced span $R \to X^2$ as in Definition 29 is a span in the category of X-premodules.

▶ **Lemma 124.** *If* Γ *is left-cocontinuous and* $\mathbf{C}$ *is locally finitely presentable and regular, then the category* $X\text{-}\mathbf{Mod}$ *is regular and the forgetful functor* $X\text{-}\mathbf{Mod} \to \mathbf{C}$ *creates all limits and colimits, as well as image factorisations.*

Proof. Creation of limits and colimits follows easily from the fact that $X\text{-}\mathbf{Mod}$ is the category of algebras for the cocontinuous endofunctor $\Gamma(-, X)$.

In particular, $X\text{-}\mathbf{Mod}$ is complete and cocomplete, hence regularity reduces to showing that regular epis are stable under pullback.

Let us first appeal to [18, §1.6.5] for definitions and preliminary results about images. Notably, in a locally finitely presentable category, (strong epi)-mono factorisations yield image factorisations, and union may be computed by cotupling followed by (strong epi)-mono factorisation.

Let us then consider any pullback square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

in $X\text{-}\mathbf{Mod}$, with f a regular epi, and show that v must also be a regular epi. By creation, hence preservation, of limits and colimits, the given pullback square is also a pullback in $\mathbf{C}$ and f is a regular epi there too. So by regularity of $\mathbf{C}$, v is a regular epi in $\mathbf{C}$. Equivalently, it is a coequaliser of its kernel pair. But by creation of limits the kernel pair uniquely lifts to a kernel pair in $X\text{-}\mathbf{Mod}$, and by creation of colimits v is a coequaliser there too. This shows that $X\text{-}\mathbf{Mod}$ is regular.

Finally, let us prove that the forgetful functor creates image factorisations. Given $A, C \in X\text{-}\mathbf{Mod}$, let us consider any image factorisation $A \xrightarrow{e} B \xrightarrow{m} C$ in $\mathbf{C}$ of a morphism $f\colon A \to C$ in $X\text{-}\mathbf{Mod}$, i.e., e is a regular epi and m is a mono in $\mathbf{C}$. In this situation, e is the coequaliser of its kernel pair in $\mathbf{C}$, but, as we just saw, this kernel pair lifts to a kernel pair in $X\text{-}\mathbf{Mod}$, whose coequaliser is created by the forgetful functor, hence e is a coequaliser, hence a regular epi in $X\text{-}\mathbf{Mod}$. Finally, f also coequalises the kernel pair, hence the existence of a unique mediating morphism $B \to C$ in $X\text{-}\mathbf{Mod}$, which must be m by faithfulness of the forgetful functor $X\text{-}\mathbf{Mod} \to \mathbf{C}$. Thus, m is also a morphism in $X\text{-}\mathbf{Mod}$. Finally, its monicity follows again by faithfulness of the forgetful functor. ◀

▶ **Lemma 125.** *If* $\mathbf{C}$ *is regular, then enhanced spans are stable under images, that is if* $p\colon R \to X^2$ *is enhanced, then so is* $\mathrm{im}(p)\colon \mathrm{im}(R) \hookrightarrow X^2$.

Proof. By Lemma 124 (creation of image factorisations). ◀

▶ **Lemma 126.** *The forgetful functor* $X\text{-}\mathbf{Mod}/X^2 \to \mathbf{C}/X^2$ *creates colimits. Hence, in particular (by cocompleteness of* $\mathbf{C}/X^2$*), enhanced spans are closed under all colimits in* $\mathbf{C}/X^2$.

Proof. Consider the following commutative diagram in $\mathbf{CAT}$.

$$\begin{array}{ccc} X\text{-}\mathbf{Mod}/X^2 & \longrightarrow & \mathbf{C}/X^2 \\ \downarrow & & \downarrow \\ X\text{-}\mathbf{Mod} & \longrightarrow & \mathbf{C} \end{array}$$

Colimits are created by both (vertical) projection functors, and also by the bottom functor by Lemma 124. Furthermore, $\mathbf{C}$ being cocomplete, the projection functor $\mathbf{C}/X^2 \to \mathbf{C}$ preserves all colimits, hence by Lemma 121 the top functor creates them. ◀

▶ **Lemma 127.** *For any syntactic signature* $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i\in n})$ *and* $X \in \sigma(\mathbf{d})\text{-}\mathbf{Trans}$*, if* $R_0 \hookrightarrow X_0^2$ *in* $\widehat{\mathbb{V}\mathbb{T}}$ *is a reflexive, enhanced flexible simulation relation, then so is* $R_0^{\mp}$.

Proof. Reflexivity is clear. For enhancedness, we have seen in Lemmas 125 and 126 that enhanced spans are closed under images and coproducts. Furthermore, closedness under span composition follows directly by Lemma 111. Finally, in order to show that $R_0^{\mp}$ is a flexible simulation, we adopt the characterisation of [18, Lemma 9.9], by which $R_0^{\mp}$ is the colimit of the chain

$$X_0 \to \mathrm{im}(R_0) \cong \mathrm{im}(R_0; X_0) \to \mathrm{im}(R_0; R_0) \cong \mathrm{im}(R_0; R_0; X_0) \to \mathrm{im}(R_0; R_0; R_0) \to \dots$$

in $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$. By Corollary 80, it suffices to show that each $\mathrm{im}(R_0^{;n})$ is a flexible simulation. By Lemma 84, it further suffices to show that each $R_0^{;n}$ is a flexible simulation. By induction and Lemma 82, it finally suffices to show that R_0 is a flexible simulation, which it is by hypothesis. ◀

Proof of Proposition 119. By hypothesis $\emptyset^{\bullet}$ is a flexible simulation. It is also enhanced by Proposition 109. Let now $R_0 := \emptyset^{\bullet\mp}$, which is again a flexible enhanced simulation by Lemma 127. By Proposition 116, R_0 is moreover symmetric. But any symmetric simulation is in fact a bisimulation, so R_0 is a flexible enhanced bisimulation. Furthermore, R_0 contains $\sim_{\mathbf{Z}}$ by Lemma 108(iv), and is a congruence by Lemma 108(i). We thus conclude by Corollary 101. ◀

A.5 The key lemma

We at last introduce the key lemma, which will directly lead us to a proof of Theorem 52.

▶ **Lemma 128.** *For any syntactic signature* $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i\in n})$*, if* Σ_1 *preserves functional flexible bisimulations, then the cartesian lifting* $\emptyset_{\mathbf{Z}}^{\bullet\Uparrow}$ *of* $\emptyset_{\mathbf{Z}}^{\bullet}$ *is a flexible simulation.*

Before proving the lemma, let us prove the main theorem, as promised:

Proof of Theorem 52. By Proposition 119, it suffices to prove that $\emptyset_{\mathbf{Z}}^{\bullet}$ is a flexible simulation, which is the case by Lemma 128. ◀

The rest of this section is a proof of Lemma 128.

► **Notation 8.** *We abbreviate* $\emptyset^{\odot}_{\mathbf{Z}}$ *to* $\emptyset^{\odot}$ *and* $\sim^{\sigma(\mathbf{d})}_{\mathbf{Z}}$ *to* $\sim$*.*

In order to prove that $\emptyset^{\bullet}_{\mathbf{Z}}$ is a simulation, it suffices to prove that $\emptyset^{\odot}_{\mathbf{Z}}$ is, by Lemma 68. Briefly, we will construct an ω-chain of flexible simulations of the form

$$\check{\Sigma}^{n}_{1}(\mathbf{Z}_0) \leftarrow R^n \rightarrow \mathbf{Z},$$

whose projection to $\widehat{\mathbb{VT}}$ is the constant chain on

$$\mathbf{Z}_0 \leftarrow \emptyset^{\odot} \rightarrow \mathbf{Z}_0. \tag{6}$$

By construction, the colimit of this chain will be a flexible simulation

$$\mathbf{Z} \leftarrow R^{\infty} \rightarrow \mathbf{Z}$$

with projection

$$\mathbf{Z}_0 \leftarrow \emptyset^{\odot} \rightarrow \mathbf{Z}_0,$$

which entails by Lemma 70 that $\emptyset^{\odot}$ is a flexible simulation as desired.

For this, let us construct a category whose objects are spans of a similar form.

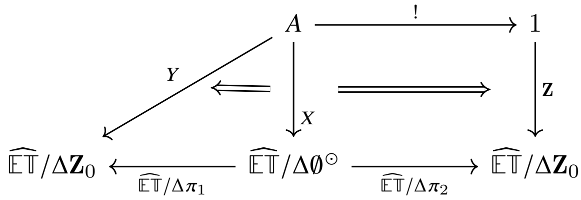
► **Definition 129.** *Let* $\mathbf{Span}/\emptyset^{\odot}$ *denote the limit of the diagram*

$$\widehat{\mathbb{ET}}/\Delta\mathbf{Z}_0 \xleftarrow{\widehat{\mathbb{ET}}/\Delta\pi_1} \widehat{\mathbb{ET}}/\Delta\emptyset^{\odot} \xrightarrow{\widehat{\mathbb{ET}}/\Delta\pi_2} \widehat{\mathbb{ET}}/\Delta\mathbf{Z}_0 \xleftarrow{\mathbf{Z}} 1$$

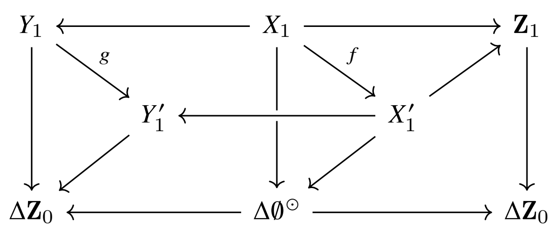
weighted by

$$2 \xleftarrow{0} 1 \xrightarrow{0} 2 \xleftarrow{1} 1$$

► Remark 130. A weighted cone from some category A is thus a diagram of the form



Hence, objects of the weighted limit are spans of the form $Y \leftarrow X \rightarrow \mathbf{Z}$ over (6), and a morphism from such a span to some span $Y' \leftarrow X' \rightarrow \mathbf{Z}$ is a pair $(g\colon Y_1 \rightarrow Y'_1, f\colon X_1 \rightarrow X'_1)$ of morphisms in $\widehat{\mathbb{ET}}$ making the following diagram commute.



► **Proposition 131.** *The forgetful functor to* $\widehat{\mathbb{ET}}^{2}$ *mapping any span* $Y \leftarrow X \rightarrow \mathbf{Z}$ *to* (Y_1, X_1) *creates colimits and connected limits.*

Proof. Straightforward. ◄

► **Proposition 132.** *The category* $\mathbf{Span}/\emptyset^{\odot}$ *has as initial object the span* $\mathbf{Z}_0 \leftarrow \emptyset^{\odot} \rightarrow \mathbf{Z}$.

Since $\mathbf{Z}_0 = \check{\Sigma}_1^n(\mathbf{Z}_0)$, this span has the desired form, and its left-hand leg is trivially a functional flexible bisimulation, so we may take it as our R^0.

► **Definition 133.** *Let F denote the endofunctor on* $\mathbf{Span}/\emptyset^{\odot}$ *that maps any object*

$$\begin{array}{ccccc} Y_1 & \longleftarrow & X_1 & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta\mathbf{Z}_0 & \longleftarrow & \Delta\emptyset^{\odot} & \longrightarrow & \mathbf{Z}_0 \end{array}$$

to

$$\begin{array}{ccccc} \Sigma_1(Y)_1 & \longleftarrow & \Sigma_1(X)_1;(\sim^*)^{\uparrow} & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,\mathrm{s,l,t}}(\Sigma_0^?(\mathbf{Z}_0),\mathbf{Z}_0) & \longleftarrow & \Delta_{\mathrm{s}}(\Sigma_0^?(\emptyset^{\odot});\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^{\odot};\mathbf{Z}_0) \times \Delta_{\mathrm{t}}(\emptyset^{\odot};\sim^*) & \longrightarrow & \Delta_{2,\mathrm{s,l,t}}(\Sigma_0^?(\mathbf{Z}_0),\mathbf{Z}_0) \\ \downarrow & & \downarrow & & \downarrow \\ \Delta\mathbf{Z}_0 & \longleftarrow & \Delta\emptyset^{\odot} & \longrightarrow & \mathbf{Z}_0. \end{array}$$

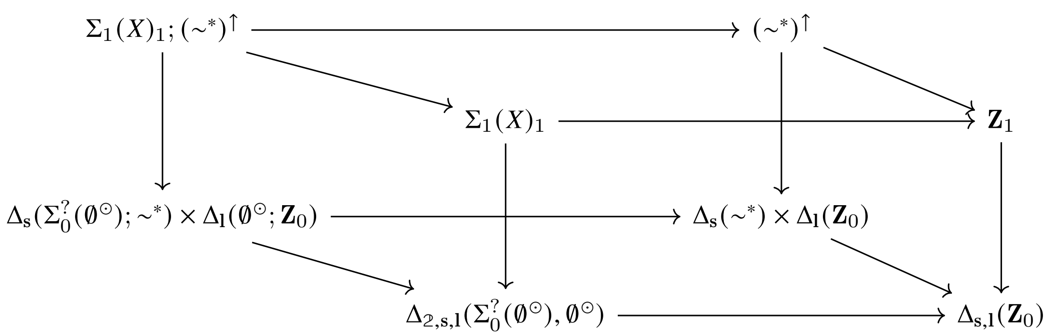
► **Definition 134.** *Let R^n be the initial F-chain.*

► **Lemma 135.** *The endofunctor F preserves flexible simulations.*

Proof. Given any span $Y \leftarrow X \rightarrow \mathbf{Z}$ over (6), the left leg of its image under F is functional flexible simulation iff the following pasting is a pointwise weak pullback.

$$\begin{array}{ccccc} \Sigma_1(X)_1;(\sim^*)^{\uparrow} & \longrightarrow & \Sigma_1(X)_1 & \longrightarrow & \Sigma_1(Y)_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{\mathrm{s}}(\Sigma_0^?(\emptyset^{\odot});\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^{\odot};\mathbf{Z}_0) & \longrightarrow & \Delta_{2,\mathrm{s,l}}(\Sigma_0^?(\emptyset^{\odot}),\emptyset^{\odot}) & \longrightarrow & \Delta_{2,\mathrm{s,l}}(\Sigma_0^?(\mathbf{Z}_0),\mathbf{Z}_0) \\ \downarrow & & & & \downarrow \\ \Delta_{\mathrm{s,l}}\emptyset^{\odot} & & \longrightarrow & & \Delta_{\mathrm{s,l}}\mathbf{Z}_0 \end{array}$$

For this, by [18, Lemma 9.26,(i)], it suffices to prove that all three inner polygons are pointwise weak pullbacks. The top right square is a pointwise weak pullback because Σ_1 preserves functional flexible bisimulations. The top left square also is a pointwise weak pullback, as the left face of the following cube.



Indeed, the top and bottom faces are pullbacks by construction, and the right face is a pointwise weak pullback because $\sim^*$ is a bisimulation. The left face being a pointwise weak pullback thus follows by [18, Lemma 9.26,(i)].

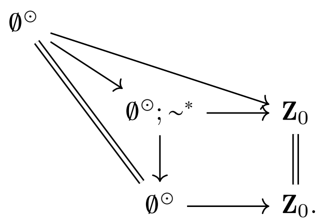
Finally, for the bottom rectangle, its domain is

$$\begin{array}{rcl} \Delta_{\mathrm{s}}(\Sigma_0^?(\emptyset^\odot);\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^\odot;\mathbf{Z}_0) & \cong & \Delta_{\mathrm{s}}(\Sigma_0^?(\emptyset^\odot);\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^\odot) \\ & \cong & \Delta_{\mathrm{s}}(\Sigma_0(\emptyset^\odot);\sim^* + \emptyset^\odot;\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^\odot) \\ & \cong & (\Delta_{\mathrm{s}}(\Sigma_0(\emptyset^\odot);\sim^*) + \Delta_{\mathrm{s}}(\emptyset^\odot;\sim^*)) \times \Delta_{\mathrm{l}}(\emptyset^\odot) \\ & \cong & \Delta_{\mathrm{s}}(\Sigma_0(\emptyset^\odot);\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^\odot) + \Delta_{\mathrm{s}}(\emptyset^\odot;\sim^*) \times \Delta_{\mathrm{l}}(\emptyset^\odot) \\ & \cong & \Delta_{2,\mathrm{s},\mathrm{l}}((\Sigma_0(\emptyset^\odot);\sim^*),\emptyset^\odot) + \Delta_{2,\mathrm{s},\mathrm{l}}((\emptyset^\odot;\sim^*),\emptyset^\odot). \end{array}$$

Similarly, we have $\Delta_{2,\mathrm{s},\mathrm{l}}(\Sigma_0^?\mathbf{Z}_0,\mathbf{Z}_0) \cong \Delta_{2,\mathrm{s},\mathrm{l}}(\Sigma_0\mathbf{Z}_0,\mathbf{Z}_0) + \Delta_{2,\mathrm{s},\mathrm{l}}(\Sigma_0^?\mathbf{Z}_0,\mathbf{Z}_0)$. The rectangle is thus obtained by applying $\Delta_{2,\mathrm{s},\mathrm{l}}$ to the (vertical) copairing of the following two squares.

$$\begin{array}{ccc} \Sigma_0(\emptyset^\odot);\sim^* & \longrightarrow & \Sigma_0\mathbf{Z}_0 \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \emptyset^\odot & \longrightarrow & \mathbf{Z}_0 \end{array} \qquad \begin{array}{ccc} \emptyset^\odot;\sim^* & \longrightarrow & \mathbf{Z}_0 \\ \downarrow & & \| \\ \emptyset^\odot & \longrightarrow & \mathbf{Z}_0 \end{array}$$

Because pointwise weak pullbacks are closed under (vertical) copairing and preserved by $\Delta_{2,\mathrm{s},\mathrm{l}}$, it thus suffices to show that both squares are pointwise weak pullbacks. The left square is one as an isomorphism in the arrow category. The right square is one because it admits a cone morphism from the actual pullback, using reflexivity of $\sim^*$ as in



◀

B Proof of Theorem 61

We assume some basic knowledge of familial functors. In particular, there is a well-known alternative characterisation in terms of **generic-free** factorisation, across which border arities are characterised as follows.

▶ **Proposition 136.** *In the setting of Definition 60, for any $\alpha \in \mathbb{ET}$ and $r \in \mathscr{U}_2'\Sigma_1(1)(\alpha)$, the border arity $\mathbf{b}_r$ is isomorphic to the morphism φ obtained by first factoring r as $\mathbf{y}_\alpha \xrightarrow{\xi_r} \mathscr{U}_2'\Sigma_1(B) \xrightarrow{\mathscr{U}_2'\Sigma_1(!)} \mathscr{U}_2'\Sigma_1(1)$ with ξ_r generic, and then $\xi_r \circ j_{2,\alpha}$ as $F(\varphi) \circ \chi_r$ with χ_r generic.*

$$\begin{array}{ccc} \mathbf{y}_{\mathrm{s}(\alpha)_D} + \sum_{i\in n_\alpha} \mathbf{y}_{(\mathrm{l}_i^\alpha)_V} & \xrightarrow{j_{2,\alpha}} & \mathbf{y}_\alpha \\ {\scriptstyle\chi_r}\downarrow & & \downarrow{\scriptstyle\xi_r} \\ \mathscr{U}_2'(\Sigma_1(A)) & \xrightarrow[\mathscr{U}_2'(\Sigma_1(\varphi))]{} & \mathscr{U}_2'(\Sigma_1(B)) \end{array}$$

▶ **Lemma 137.** *For any categories with generating cofibrations* $(\mathscr{A}, \mathbb{J})$ *and* $(\mathscr{B}, \mathbb{K})$*, a familial functor* $F \colon \mathscr{A} \to \mathscr{B}$ *preserves fibrations iff it is* ***cellular****, in the sense that for all commuting squares*

$$\begin{array}{ccc} C & \xrightarrow{k} & D \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \chi} \\ F(X) & \xrightarrow[F(\delta)]{} & F(Y) \end{array} \tag{7}$$

with $k \in \mathbb{K}$ *and* ξ *and* χ *generic,* δ *is a cofibration (i.e.,* $\delta \in {}^{\pitchfork}(\mathbb{J}^{\pitchfork})$*).*

Proof. This is a straightforward generalisation of [18, Lemma 7.28], whose proof applies *mutatis mutandis.* ◀

Proof of Theorem 61. We assume given a dynamic signature $\Sigma_1 \colon \sigma\text{-}\mathbf{Trans} \to \sigma\text{-}\mathbf{Trans}_2$ such that $\mathscr{U}_2\Sigma_1$ is familial. By that that By Proposition 59, Σ_1 preserves functional flexible bisimulations if and only if $\mathscr{U}_2\Sigma_1$ maps $\mathbb{J}_\sigma$-fibrations to $\mathbb{J}_{2,\sigma}$-fibrations, or equivalently, by Lemma 137, if it is cellular.

Clearly, $\mathscr{U}_2\Sigma_1$ is cellular, then the border arities of all rules are $\mathbb{J}_\sigma$-cofibrations, by a straightforward instantiation of Diagram 7. Conversely, assume that all rules are $\mathbb{J}_\sigma$-cofibrations and consider a commuting square as in Diagram 7, taking $F = \mathscr{U}_2\Sigma_1$, specialised to the involved sets of cofibrations:

$$\begin{array}{ccc} \mathbf{y}_{\mathrm{s}(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(1^\alpha_i)_V} & \xrightarrow{j_{2,\alpha}} & \mathbf{y}_\alpha \\ {\scriptstyle \chi_r}\downarrow & & \downarrow{\scriptstyle \xi_r} \\ F(A) & \xrightarrow[F(\varphi)]{} & F(B). \end{array}$$

The result follows by considering the rule $\mathbf{y}_\alpha \xrightarrow{\xi_r} F(B) \xrightarrow{F(!)} F(1)$ and exploiting uniqueness (up to isomorphism) of generic factorisations [18, Remark 7.19]. ◀