Types are weak omega-groupoids, in Coq

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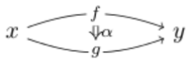
The problem: types and weak ω-groupoids, internally One of the discoveries underlying the recent homotopical interpretation of Martin-Löf type theory is the fact that, for any type, the tower of its iterated Martin-Löf *identity* types gives rise to a weak ω-groupoid [LL10, vdBG11]. Some motivation has recently been put forward [ALR14, HT15] for performing this construction *internally*. However, first investigations have stumbled upon the following tension: (1) on the one hand, in order to semantically agree with the standard notion, ω-groupoids should be based on *sets*, i.e., 'discrete' types; (2) on the other hand, constructing the weak ω-groupoid associated to any non-discrete type appears to require basing them on general types.

The latter constraint is clear, but let us give a bit more detail about the former. As previous authors, we adopt Brunerie's definition of weak ω-groupoids [Bru16], which seems most directly amenable to internalization. Roughly, a weak ω-groupoid is a model (in types) of a specific, very simple type theory. In the most basic version of this theory, contexts correspond to finite globular sets, types to pairs of parallel cells, and terms to cells. Thus, for example, the globular set below left is modeled by the context below right



$$x : \star, y : \star, f : x =_\star y, g : x =_\star y, \alpha : f =_{x =_\star y} g \vdash .$$

Example types in it are $f =_{x=_\star y} g$ and $\alpha =_{f =_{x=_\star y} g} \alpha$; and an example term of the former type is α itself. In this basic version, models of the theory are merely globular types. The idea is then to enrich the type theory with term formers corresponding to composition, identities, associativity, and the whole standard package of higher coherences. This is done by first identifying a class of *contractible* contexts which correspond to pasting schemes. E.g., pasting schemes for binary composition and identities for 1-cells are given by

$$x : \star, y : \star, f : x =_\star y, z : \star, g : y =_\star z \vdash \qquad \text{and} \qquad x : \star \vdash .$$

Such contexts may be defined inductively, which yields a different judgement $\Gamma \vdash_c$. The crucial rule then (roughly) says that any type A in any contractible context Γ in inhabited by a term $\mathbf{coh}_{\Gamma,A}$, called a *coherence,* which amounts to saying that the corresponding pasting scheme admits a composite. E.g., composition is the coherence obtained for the context above left, with $A = (x =_\star z)$, and identity corresponds to the one above right with $A = (x =_\star x)$.

Now the difficulty evoked in item (1) above arises in the definition of models of this type theory, which is polluted with coherence conditions. Typically, a naïve approach could start by defining a model to consist of a type $[\![\Gamma]\!]$ for each context Γ, plus for each $\Gamma \vdash A$ a family $[\![A]\!]_\Gamma$ indexed by the elements of $[\![\Gamma]\!]$, and for each term $\Gamma \vdash M : A$ a section of this family. But for any $\Gamma \vdash B$, there is another family $[\![A]\!]_{\Gamma,B}$, and we certainly want any model to satisfy the condition that the latter is precisely given by $(\gamma, b) \mapsto [\![A]\!]_\Gamma(\gamma)$, for any $\gamma \in [\![\Gamma]\!]$ and $b \in [\![B]\!]_\Gamma(\gamma)$. Such constraints are hard to specify without any redundancy. They thus generate higher constraints, and so on, which quickly becomes intractable.

A possible solution [ALR14] consists in taking models in *sets,* i.e., types with discrete homotopy type, which considerably improves the situation but gives up item (2).

Two-level type theory We here avoid this dilemma, by working in a *2-level* type theory [ACK16], i.e., a type theory with two notions of equality, one *strict* and one *homotopical.* We formalize the construction in a simple 2-level extension of Coq [Laf]. The point is to use strict equality to axiomatize weak ω-groupoids: the constraints evoked above are required to hold strictly, which has the same taming effect as taking models in sets, while still accomodating models in general types. Homotopical equality may then be used to construct the weak ω-groupoid associated to a so-called *fibrant* type. This idea turned out to work, but only up to the following issues, which are arguably of lesser conceptual importance.

First, the construction of the weak ω-groupoid associated to a type is rejected by Coq's well-foundedness criterion. We thus consider a variant of Brunerie's type theory which only considers contractible contexts, and accordingly terms and types therein. This is enough for Coq to swallow the pill, but of course we should adapt our notion of model to compensate for the missing contexts. However, non-contractible contexts never yield new coherences, hence only have to do with globular structure, not weak ω-groupoid structure. We may thus define weak ω-groupoids as models of our restricted type theory *in globular types,* so that globular structure is built into them from the start.

A second issue is related to defining type theories internally. In [ALR14], the authors formalize Brunerie's type theory as an *intrinsic* syntax, i.e., as an inductive-inductive-recursive datatype following the typing rules directly, although they have to switch off Agda's termination checker. However, Coq does not support such definitions, so we follow the old school route: we define untyped syntax first and then typing judgements, and use the Uniqueness of Identity Proofs principle satisfied by strict equality to simulate the non-dependent inductive-inductive-recursive eliminator required to construct the weak ω-groupoid associated to a type.

Weak ω-categories Finally, slightly restricting contractible contexts and the coherence rule, we also formalize weak ω-categories of [FM17]. We leave the following consistency check for future work: does any Finster-Mimram ω-category with weakly invertible cells yield a Brunerie ω-groupoid?

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