# A short note on Initial Algebra Semantics for QIITs

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**Abstract**

[ACDF16] provides a setting for Initial Algebra Semantics for Quotient Inductive Inductive types and conjectures an initiality result given adequate accessibility conditions. We make these conditions precise and proves the conjecture, by showing that the categories of models are actually locally presentable and thus cocomplete.

## 1 Preliminaries

We introduce some standard results about *locally presentable categories*. A standard reference is [AR94].

No need to know the exact definitions; we give below the important examples. Let us just quickly mention that to show that a functor is accessible, one must first ensure that the domain and codomain categories are accessible and that the functor preserves some specific shapes of colimits.

**Example 1.1.** *[AR94, Examples 1.12 and 2.3.(1)] Any presheaf category is accessible. In particular, the category* $\mathrm{Set} \cong [1, Set]$ *of sets and the terminal category* $1 \cong [\emptyset, \mathrm{Set}]$ *are accessible.*

**Example 1.2.** *The identity endofunctor on an accessible category is accessible.*

**Lemma 1.3.** *[AR94, Remark 2.18.(2)]The composition of two accessible functors is accessible.*

The main result that we will need is the following.

**Lemma 1.4.** *If a complete category is accessible, then it is cocomplete (thus it has an initial object). Such a category is called* locally presentable.

*Proof.* By [AR94, Corollary 2.47]. □

[ACDF16] proves completeness of the categories of models. Therefore, the only thing to ensure is that they are accessible. To this end, we will find an equivalent definition of the category of algebras using the following standard constructions.

**Lemma 1.5.** *[AR94, Theorem 2.43] If $F : C \to D$ and $G : C' \to D$ are accessible functors, then so is the comma category $F/G$ .*

**Definition 1.6.** *[AR94, 2.71] The* inserter *of two functors $F, G : C \to D$ is the category defined as follows: an object is an object $c$ and a morphism $Fc \to Gc$; a morphism between $Fc \to Gc$ and $Fc' \to Gc'$ is a morphism $c \to c'$ making the obvious square commute.*

**Definition 1.7.** *[AR94, 2.76] The* equifier *of two natural transformations $\alpha, \beta$ between functors $F, G : C \to D$ is the full subcategory of $C$ consisting of objects $c$ such that $\alpha_c = \beta_c$.*

**Lemma 1.8.** *[AR94, Theorem 2.72] If $F, G : C \to D$ are accessible, then their inserter is accessible.*

**Lemma 1.9.** *[AR94, Lemma 2.76] If $F, G : C \to D$ are accessible, then the equifier of any pair of natural transformations $\alpha, \beta : F \to G$ is accessible.*

## 2 Specification of sorts

**Lemma 2.1.** *Given a specification of sorts [ACDF16, Definition 3] as a list $H_0, H_1, \ldots, H_{n-1}$ of functors $H_i : C_i \to \mathrm{Set}$, each category $C_{i+1}$ is equivalent to the comma category $\mathrm{Set}/H_i$.*

*Proof.* Straightfoward. ☐

**Corollary 2.1.1.** *If each $H_i$ is accessible, then each $C_i$ is accessible.*

*Proof.* $C_0$ is accessible by Lemma 1.1. Moreover, $C_{i+1} \cong Set/H_i$ is accessible by Lemma 1.5, Examples 1.2 and 1.1. ☐

**Corollary 2.1.2.** *If each $H_i$ is accessible, then each $C_i$ is locally presentable.*

*Proof.* By Lemma 1.4, because each $C_i$ is complete by [ACDF16, Theorem 10]. ☐

## 3 Constructor specifications

**Remark 3.1.** *As explained by the paragraph below [ACDF16, Example 18], a* constructor specification *[ACDF16, Definition 12] on a complete category $C$ is equivalently given by two functors $F, G : C \to \mathrm{Set}$ and a natural transformation $\alpha : G \to F$ such that an additional property stated by [ACDF16, Lemma 19] holds (this property ensures that the category of algebras is complete).*

**Lemma 3.2.** *Given a constructor specification as above, the category of algebras is equivalent to the category defined as follows: an object is an object $c$ of $C$ equipped with a section of $Gc \to Fc$; a morphism is a morphism in $C$ that makes the obvious square commute.*

*Proof.* Straightforward. □

**Corollary 3.2.1.** *If $F$ and $G$ are accessible, then the category of algebras is accessible.*

*Proof.* Let $D$ be the inserter of $F$ and $G$. It is accessible by Lemma 1.8. Now consider the functor $F' : D \to \mathrm{Set}$ mapping $Fc \xrightarrow{f} Gc$ to $Fc$. Consider the natural transformations $\alpha', \beta' : F' \to F'$ whose components at $Fc \xrightarrow{f} Gc$ are respectively given by $Fc \xrightarrow{f} Gc \xrightarrow{\alpha_c} Fc$ and $Fc \xrightarrow{id_{Fc}} Fc$. Then, by Lemma 3.2, the category of algebras is equivalent to the equifier of $F$ and $G$. By 1.9, it is enough to show that $F'$ is accessible. Now, note that $F'$ is the composition of $D \to C \xrightarrow{F} \mathrm{Set}$. By assumption, the second functor is accessible. By Lemma 1.3, it is enough that the forgetful functor $D \to C$ is accessible. But this is precisely [AR94, Remark 2.73]. □

**Corollary 3.2.2.** *If $F$ and $G$ are accessible, then the category of algebras is locally presentable.*

*Proof.* By Lemma 1.4, because the category of algebras is complete by [ACDF16, Theorem 27]. □

# 4 Conclusion

[ACDF16] constructs the final category of models of a QIIT in two stages: first using a sort specification, and then a hierarchy of constructor specifications. Corollary 2.1.2 ensures that the first stage yields a cocomplete category, while Corollary 3.2.2 ensures that the second stage finally yields a cocomplete category. Therefore, the existence of the initial object follows.

# References


[ACDF16] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, and Fredrik Nordvall Forsberg. Quotient inductive-inductive types. *CoRR*, abs/1612.02346, 2016. URL: `http://arxiv.org/abs/1612.02346`, `arXiv:1612.02346`.

[AR94] J. Adámek and J. Rosicky. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994. `doi:10.1017/CBO9780511600579`.